Universidade Federal de Alfenas
Campus Poços de Caldas – Poços de Caldas - MG

BRUNA MUNIZ PERES

Aharonov-Bohm Effect in the Podolsky's Generalized Electrodynamics

Poços de Caldas/MG
2021

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Dissertação apresentada como parte dos requisitos para obtenção do título de Mestre em Física pelo Programa de Pós- Graduação em Física da Universidade Federal de Alfenas. Área de concentração: Física de Partículas e Campos.
Orientador: Prof. Dr. Cássius Anderson Miquele de Melo.

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Peres, Bruna Muniz.
Aharonov-Bohm Effect in the Podolsky's Generalized Electrodynamics / Bruna Muniz Peres. - Poços de Caldas, MG, 2021.
63 f. : il. -

Orientador(a): Cássius Anderson Miquele de Melo.
Dissertação (Mestrado em Física) - Universidade Federal de Alfenas, Poços de Caldas, MG, 2021.
Bibliografia.

1. Efeito Aharonov-Bohm. 2. Eletrodinâmica de Podolsky. 3. Eletrodinâmica de Maxwell. 4. Eletrodinâmica de ordem superior. 5. Simetria de calibre. I. de Melo, Cássius Anderson Miquele, orient. II. Título.

Ficha gerada automaticamente com dados fornecidos pelo autor.

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Aprovada em: 30 de abril de 2021.

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Dedico este trabalho,
aos meus avós que sempre me incentivaram
a ir atrás dos meus sonhos.

Acknowledgements

First and foremost, I would like to thank my family, my paternal grandparents, Cevandyra *(in memoriam)* and Heraldo, and my father, Heraldo Junior, for all the support, love, and care they have always given me. To Regise and João Pedro, for adopting me as family. I am especially grateful to my grandmother Cevandyra, who provided me with the opportunity to take an English course that made the writing of this work possible, and who, in our conversations, always brought me peace. I also thank Idalina and Zoraide, whom I affectionately call ”aunts,” and who always motivated me to keep studying.

To my boyfriend Eduardo, for his patience every time I couldn't stop talking about the master's program when I should have been resting. To his parents, Rosangela and Luís, for always welcoming me so warmly, and to Raul, for the small talks.

To my friends from Itapira and Poços de Caldas: Jéssica, Vanilda, Paulo Vitor, Lucas, Natália, Tayná, Arthur, Michael, and Walter, for also putting up with me talking about my project even though no one understood anything. To Carol, who was my first friend in Poços at the apartment, and to Leo, who made our apartment a cozier place with the smell of coffee every morning.

To my master's colleagues: Silas, ”Cado,” Carlos, Wallison, Yuri, Juan, Hiago, Lucas, and Christiane, for all the hours of study, beers, rides, and advice. You were essential in making the master's program more enjoyable.

To my advisor, Professor Cássius, for all the years and all the times he was more than a dissertation advisor, but also my psychologist in the moments when I felt that nothing would work out. To Professor Rodrigo, for believing in me when I first became interested in Physics during my undergraduate research. To the program's professors: Gardim, Gustavo, Alencar, and Vivas, for their teachings, availability, and the lively atmosphere in the hallways. To Professor Laos, for guiding me to the master's program during my tutoring orientation, one of my inspirations for choosing this path.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

To everyone who was involved at some point in this work and whom I may have forgotten, and to the reader who is interested in the subject, my heartfelt thanks.

"For whatever reason, I didn't succumb to the stereotype that science wasn't for girls. I got encouragement from my parents. I never ran into a teacher or a counselor who told me that science was for boys. A lot of my friends did."

- Sally Kristen Ride [1]

[1] Speech given in an interview with USA TODAY, in the United States, on March 19, 2006

ABSTRACT

Maxwell's electromagnetism rewrote the four experimental laws of classical electrodynamics by combining fields, sources, charges, and electricity. If we propose a Lagrangian analysis of Maxwell's theory with higher-order derivatives, making it also free of gauges, we obtain the so-called Podolsky electrodynamics. As a higher-order electrodynamics, Podolsky's electrodynamics is studied as a generalization of Maxwell's theory, becoming a specific case of Podolsky when the free parameter is zero. From these theories, we will study the magnetic effect of Aharonov-Bohm, where electric charges are affected by the magnetic field through its potential vector in a solenoid. In classical theory, the magnetic field should not exist in the region where the charges pass; only the potential could cause such an alteration. The result is differences in electrons, particularly in their phases in quantum mechanics between wave functions, in a specific amount called the Aharonov-Bohm phase. In this thesis, the Aharonov-Bohm effect is studied for the Maxwell case and the Podolsky case, comparing the difference that a potential vector can make in these results and the phase effect even in areas without a magnetic field.

Keywords: Aharonov-Bohm effect. Podolsky electrodynamics. Maxwell electrodynamics. Higher-order electrodynamics. Gauge symmetry.

RESUMO

O eletromagnetismo de Maxwell reescreveu as quatro leis experimentais da eletrodinâmica clássica combinando campos, fontes, cargas e eletricidade. Se propusermos uma análise lagrangiana da teoria de Maxwell com derivadas de ordem mais alta, tornando-a também livre de calibres, obtemos a chamada eletrodinâmica de Podolsky. Como eletrodinâmica de ordem superior, a eletrodinâmica de Podolsky é estudada como uma generalização da teoria de Maxwell, tornando-se um caso específico de Podolsky quando o parâmetro livre é zero. A partir destas teorias, estudaremos o efeito magnético de Aharonov-Bohm, onde as cargas elétricas são afetadas pelo campo magnético pelo seu potencial vetor num solenoide. Na teoria clássica, o campo magnético não deveria existir na região onde as cargas passam, apenas o potencial poderia causar tal alteração. O resultado são diferenças nos elétrons, particularmente nas suas fases na mecânica quântica entre funções de onda, numa quantidade específica chamada fase Aharonov-Bohm. Nesta tese, o efeito Aharonov-Bohm é estudado para o caso Maxwell e o caso Podolsky, comparando a diferença que um potencial vetor pode fazer nestes resultados e o efeito de fase mesmo em áreas sem um campo magnético.

Palavras-chave: Efeito Aharonov-Bohm. Eletrodinâmica de Podolsky. Eletrodinâmica de Maxwell. Eletrodinâmica de ordem superior. Simetria de calibre.

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1 INTRODUCTION

At the end of the 19th century, electromagnetism had great discoveries and transformations by James Clerk Maxwell, who came up with the four experimental laws of classical electrodynamics combining fields, sources, electric charges and currents. Years later, with the development of relativity and quantum physics, the pursuit of a unique equation that could explain electromagnetic behavior and all physical phenomena, many studies and experiments were made, and conclusions like the nature of light and its speed were made .

In 1959, Aharonov and Bohm came up with an ansatz solution for Schrödinger's equation that could reaffirm the use of potentials with physical purposes [1]. The experiment was done with a high collimated beam of electrons passing on both sides of a long current-carrying solenoid and recombined at the end. The result was a difference in the electrons, specifically in its quantum mechanical phase shift between wave functions [2].

The studies by Aharonov and Bohm started with magnetism, specifically the magnetic potentials, which until today have no proof of physical existence. His theoretical experiments involved electric charges passing through magnetic fields, in order to verify whether they would be affected by the field [1]. In classical theory, the field should not exist in the region where the charges would pass, that is, outside a solenoid. They realized then, that the charges were affected in their quantum phase, in a specific amount that was called the Aharonov-Bohm phase. This phase showed that the charged particle suffered interaction with the potential since the field was null, but the potential could not be.

Years later, Aharonov continued his studies, this time testing for electricity, along with Casher. The new experiment consisted of an alternative to the first experiment: instead of an electrically charged solenoid, it was magnetized and therefore the field produced was an electric field, and magnetized particles would pass around it. With the similarity in the experiments, the results were similar and the difference in the phase was also noticed in the arrival of the particles [3].

In this thesis, the magnetic Aharonov-Bohm effect was studied for the Maxwell case and the Podolsky case, comparing the differences that the potential vector could make in those results and if it influences the phase even in a region with no magnetic field.

Maxwell's equations are a model that describes well the electromagnetic phenomena of classical physics. While a pair of Maxwell's equations can be reached by Maxwell's Lagrangian, the others are obtained from the invariant gauge that imposes a condition of dependence between the electric and magnetic fields. If we propose an alteration to the Lagrangian, we would obtain Podolsky's electrodynamics equations. These equations do not preserve the symmetry between the magnetic and electrical properties according to Maxwell's equations.

In higher-order electrodynamics, Bopp-Podolsky's [4] electrodynamics is studied as a generalization of Maxwell's electrodynamics, which becomes a specific case of Podolsky when the free parameter is null. In recent studies, this theory is been used to study other subjects such as the problem of the duality of electromagnetic theory[5], the relationship with gravitation and black holes [6] and applications in cosmology [7].

Podolsky's electrodynamics have an effective Lagrangian, which comes from a correction in Maxwell's Lagrangian [8]. In this theory, we see that the addition of the mass in the Lagrangian does not oppose the gauge symmetry, as in Proca's Lagrangian [9]. The first estimate for the Podolsky parameter was given in [10], while the most recent can be found as $m_P \geq 3{,}7595.10^{10} eV$, by Bhabha scattering in the range of $12GeV \leq \sqrt{s} \leq 46.8GeV$ [11].

From Podolsky we can also have explanations about the Abraham-Lorentz factor, where the 4/3 appear when the calculation is made with the electromagnetic mass, fixed in this electrodynamics [12]. We know that the theory works for high energies [**?**] and that vacuum self-energy is finite [13].

In this work, we will show how the Aharonov-Bohm effect, calculated using Podolsky's electrodynamics, can show information about the magnetic potential that does not appear in classical electrodynamics.

2 THEORETICAL FOUNDATION

2.1 MAXWELL'S ELECTRODYNAMICS

As seen in the Introduction, electromagnetism had been through great discoveries and transformations by the years since its discovery, until Maxwell came up with the four experimental laws of classical electromagnetism combining fields, fonts, electric charges, and currents.

Before Maxwell's studies, the nature of light was somehow mysterious and not truly understood, once it showed itself as light, and at other times it behaved like a particle. Thanks to Maxwell, nowadays it is common to say properly that light is a type of electromagnetic wave and among other conclusions, the speed of light can be calculated through Maxwell's equations, presented below as integrals:

$$\begin{cases} \oint\limits_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{in}}{\epsilon_0} \\ \oint_S \boldsymbol{B} \cdot d\mathbf{A} = 0 \\ \oint_C \boldsymbol{B} \cdot d\mathbf{l} = \mu_0 \left(I + \epsilon_0 \frac{d\phi_{\mathbf{E}}}{dt} \right) \\ \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi_{\mathbf{B}}}{dt} \end{cases} \tag{2.1}$$

The first equation is that the Gauss law concerning the electric flux passing through any closed surface is equal to $1/\epsilon_0$ multiplied by its charge in the interior of the surface. This implies that the electric field E is generated by a punctual charge that varies inversely with the square of the distance to the charge; thus this law describes how the electric field lines come out from the positive charge and converge to the negative charge. It is based on the experimental Coulomb law.

Gauss law for the magnetism, the second equation, says that the flux of the magnetic field B through any closed surface is zero, describing the experimental observation that the magnetic field lines do not diverge from anywhere in space nor converge to any point in space, implying that isolated magnetic poles do not exist.

The third equation, the Faraday law, explains that the circulation of the electric field E around any closed curve C is equal to the negative rate of change of the magnetic field flux B through any surface S limited by the curve C. If S is not a closed surface, the magnetic flux through S is not necessarily zero. Describes how the electric field lines circle any area where the magnetic flux varies and relates the electric field vector E to the rate of change of the magnetic field vector B.

Ampere law, presented in the last equation, is modified so it can include Maxwell displacement current, says that the line integral of magnetic field B around any closed surface S bounded by the curve and the displacement current across the same surface, describing how the field lines circle from end to end by a current or displacement current passing.

$$\begin{cases} \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \\ \nabla \cdot \boldsymbol{B} = 0 \\ \nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \mathbf{J} \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0 \end{cases} \tag{2.2}$$

We can observe that the first and third equations have sources and come from a Lagrangian description. Meanwhile, the second and the fourth equations guarantee the existence of the vector potentials **A** and **V** plus the gauge symmetry, and being homogeneous equations, don't have sources nor Lagrangians, since the Maxwell Lagrangian is:

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu \tag{2.3}$$

in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$.Knowing that A_ν is four-vector potential and that $F^{\mu\nu}$ is the Faraday tensor and the metric η has signature $(+, -\ , -\ , -)$.

2.2 PODOLSKY'S ELECTRODYNAMICS

An interesting generalization of the usual electrodynamics was developed by Boris Podolsky in 1942 [4, 13]. It is an extension of Maxwell's theory, in which a term involving the second-order derivative of the electromagnetic field is added to Maxwell's Lagrangian. Podolsky's electrodynamics shows that this theory is the only second-order caliber-invariant generalization possible for Maxwell's electrodynamics, making this theory an alternative for the study of electromagnetic phenomena. The equations are shown next using the d'Alembertian ($\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$) and can turn back to Maxwell using $a = 0$.

$$\begin{cases} (1 - a^2\Box) \nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \\ \nabla \cdot \boldsymbol{B} = 0 \\ (1 - a^2\Box) \left(\nabla \times \boldsymbol{B} - \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) = \mu_0 \mathbf{J} \\ \nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \mathbf{0} \end{cases} \tag{2.4}$$

The Lagrangian in Podolsky electrodynamics is given by:

$$\mathcal{L}_P = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu F^{\mu\nu} \partial_\zeta F^\zeta_\nu + A_\mu J^\mu \tag{2.5}$$

in which $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$.

Podolsky's additional term is the only possible term for an effective second-order linear theory, compatible with gauge symmetry [9]. These additional terms also have some similarities with the behavior of a plasma [14], which can be explained by a one-loop effective theory of higher order [8]. Podolsky theory was used in the study of various quantum aspects [**?**, **?**, 11, 15, 16], however, in this work we will study how the classical Podolsky field can impact the quantum behavior of charged particles.

3 AHANOROV-BOHM EFFECT IN MAXWELL ELECTRODYNAMICS

The Aharonov-Bohm effect can be simply resumed as a solenoid in which charged particles pass nearby and undergo changes in phase due to its magnetic field [17]. It can be solved with some steps necessary for Maxwell electrodynamics. At first, it is necessary to know if there is a magnetic field at all and, if so, where it is located. To do so, the Ampère law is used, calculating the magnetic field inside and outside the solenoid. Knowing these results, we can also calculate the magnetic vector potential for inside and out of the solenoid, and then, apply the potential for outside the solenoid (where the effect is shown) in Schrödinger's equation.

Solving the Schrödinger equation with the external potential calculated earlier, the phase and quantum energy are achievable and, when possible, compared with the experimental results. These last results are the most important for the Aharonov-Bohm effect, detectable by the phase variation and, when using Maxwell's electrodynamics, it also proves that the vector potentials may not be just a good mathematical tool to reach the magnetic field we are interested in , but also something physically real. Starting the calculation script, in the next sections we will see how the diffraction patterns are presented and how to calculate the magnetic field in a long solenoid.

3.1 DIFFRACTION PATTERNS

The experimental setup of the classic two split diffraction experiment involving an electron beam can be seen in the Figure 1.

The intensity pattern is given as a function of the θ angle and the distance δ of the slit [18].

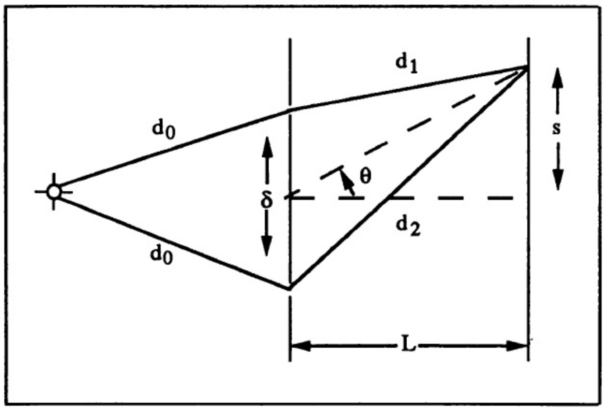
$$I_\theta = 4I_0 cos^2 \left(\pi\delta \frac{sin\theta}{\lambda} \right) \tag{3.1}$$

The result is shown in Figure $2(a)$ forming the electron diffraction pattern [18]. Changing the assembly can be made by placing an infinite solenoid behind the bulkhead between the slots, as seen in Figure $2(b)$.

Since the solenoid is infinite, there is no magnetic field in the region where the beam passes through. With this assembly, the diffraction pattern is shifted, and the intensity gains a term, as we can see in equation [18]:

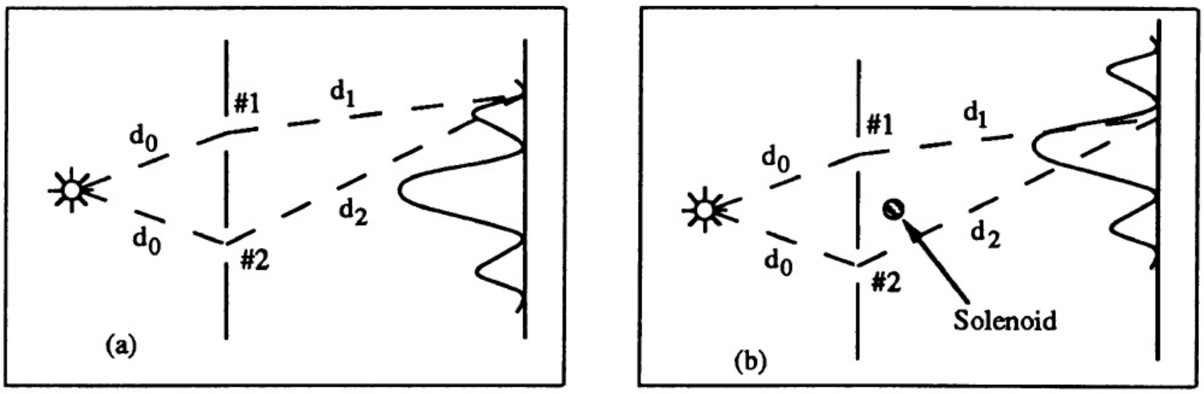
$$I_\theta = 4I_0 cos^2 \left(\pi\delta \frac{sin\theta}{\lambda} + e\frac{1}{2}B\pi R^2 \right) \tag{3.2}$$

Figure 1 – Geometry of the two split diffraction.



Source:[18].
Caption: Definitions of the measures in the experiment.

Figure 2 – Different assembles of the experiment.



Source: [18]
Caption: In the Figure (a) the classic result is shown, while in the Figure (b) there is also a solenoid.

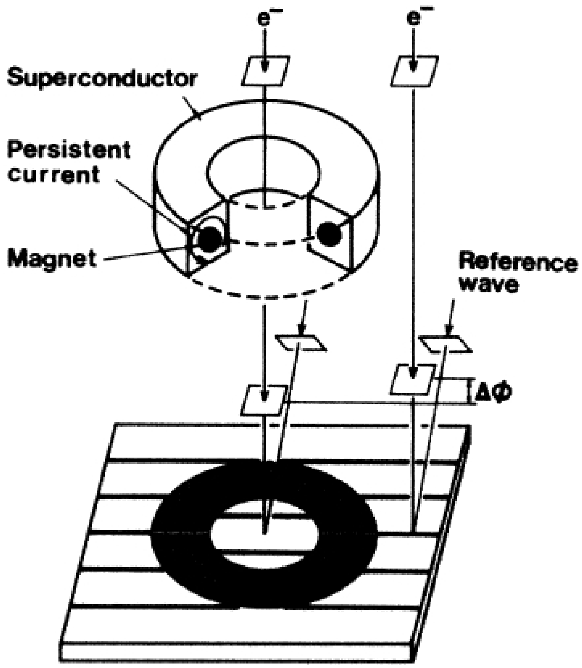
In 1986 an experimental setup was made by Tonomura et al [19] with a toroidal magnet, where an electron beam passes inside the toroid and one outside, as can be seen in the image 3.

The images that were generated can be seen in figure 4, where the white lines are the crest of the wave and the black lines are the valley. The beam intensity was adjusted to 2π, to ensure that the phase reverses when it passes through the toroid [19].

In the figure 5, we can see in detail the displacement of the lines [20].

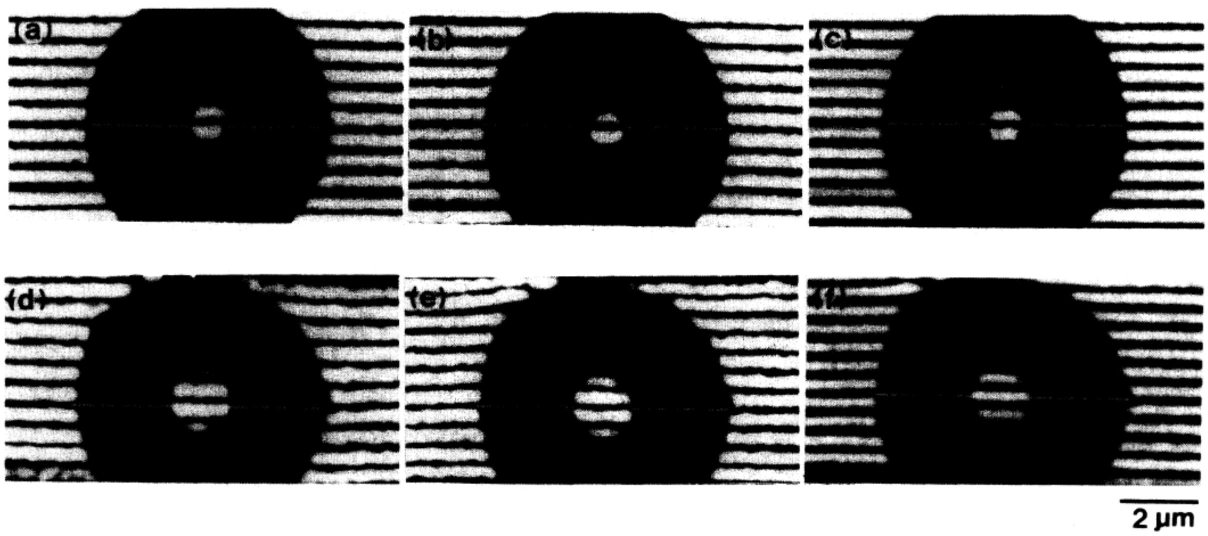
That amount of displacement would then be the phase displacement in Aharonov-Bohm, as seen in details next.

Figure 3 – Tonomura's experimental setup.



Source: [19].
Caption: Experimental setup using a toroidal magnet.

Figure 4 – Diffraction patterns.

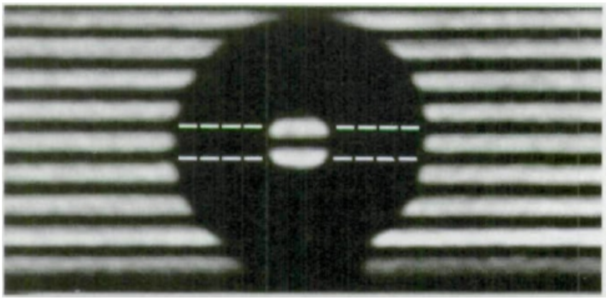


Source: [19].
Caption: Images (a) to (f) shows the different patterns in the experiment.

3.2 MAGNETIC FIELD IN A LONG SOLENOID

A solenoid is a conductive wire wrapped in a helix with turns very close together. It can produce an intense and uniform magnetic field nearby its rings. Essentially, the magnetic field of a solenoid is the set of n identical rings placed side by side. A solenoid is considered long when its length L is much longer than its radius r. In the interior of a long

Figure 5 – Displacement of lines.



Source: [20].

Caption: The dashed line indicates the continuity of the wave that may have been moved up or down.

solenoid and away from the edges, the effect of the radius on each side are complementary, so the magnetic field is represented by parallel lines from the axis and are close and evenly distributed, since the magnetic field is intense and uniform. Besides, the field lines separate when we move away from both sides of the solenoid (above and below) since there's no magnetic field influence further from the radius.

When studying the Aharonov-Bohm effect an experimental setup is needed, using a solenoid and a high collimated beam. The charged solenoid will generate a magnetic field inside it, then the beam is divided into two, surrounding the solenoid and recombined after. So, the logical steps for studying all the effects are, first of all, to find the magnetic field that is generated by the solenoid. Knowing the magnetic field, we can calculate the magnetic vector potential, outside the solenoid, where the effect is shown. Then, we can use the magnetic vector potential to calculate Schrödinger's equation for the wave equation, used to calculate, later, the energy and the phase difference, which is the main target of the study. Those calculations are done first for classical electrodynamics, using Maxwell's equations and in the possession of those results, recalculated using high order electrodynamics, using Podolsky's equations. Starting this script, the magnetic field by Ampére's law is shown next.

3.3 AMPÈRE'S LAW

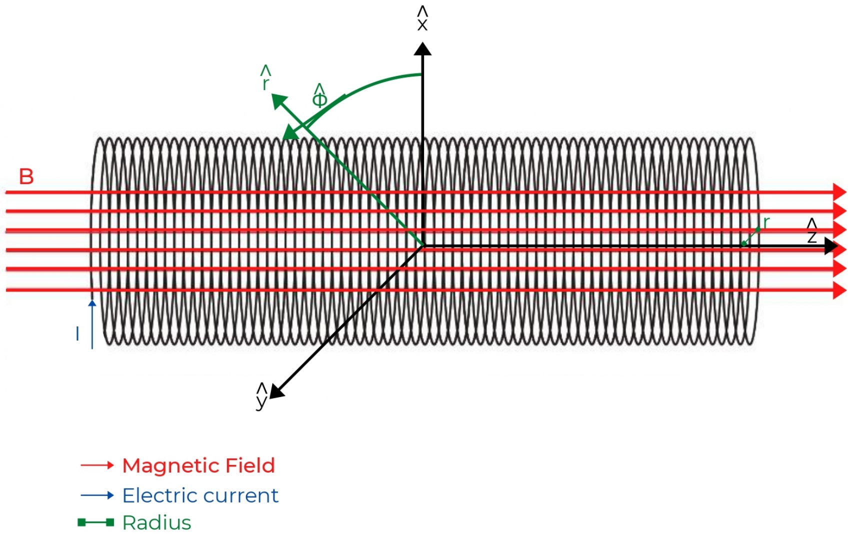
Ampère's law relates the tangential compound of the total magnetic field **B** integrated along with a closed curve C to the current I that passes through any surface bounded by C. This can be used to calculate the magnetic field of high-level symmetry systems.

Mathematically:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{3.3}$$

where μ_0 is the magnetic permeability of free space. In the Figure 6, a charged

Figure 6 – A charged solenoid.



Source: Author (2021).
Caption: When the solenoid is charged, a magnetic field is generated in its radial direction.

solenoid with radius r and charged with an electric current I, generating the magnetic field $\mathbf{B}$.

The tangential direction is given so the choice of the positive flux of the current I through the S path. Ampère's law is only valid while the current is constant and continuous, so there is no time variation and no cumulative charges anywhere. By the high symmetry, the line integral 3.3 can be written as the product $\mathbf{B}$ by some distance. Ampère's and Gauss's laws have high theoretical importance and both are useful if there is symmetry. If there is not, none of them are practical to calculate the electric or magnetic field.

A solenoid can be figured as a collection of rings, centered and piled up. When an electric current I flows through it, a magnetic field is generated. So for a ring with radius r, the magnetic field is calculated by 3.3. Considering that the solenoid has n rings when divided by its length l, we can define the density of spires at each unit of solenoid length as N.

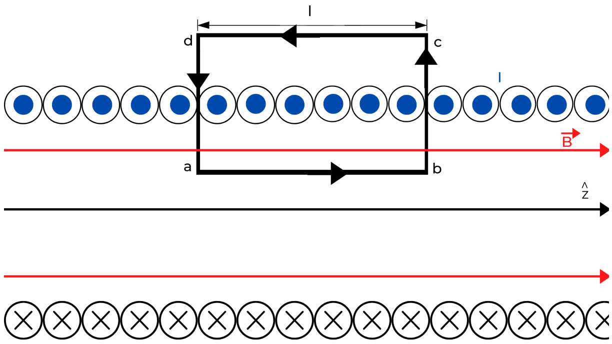
Drawing an amperian surface, which goes from "a" to "d" on a rectangular path, the magnetic field can be calculated as a summation of all parts of the path, from the center of the ring, as shown in Figure 7.

In the formula,

$$\oint \mathbf{B} \cdot d\mathbf{r} = \int_a^b \mathbf{B} \cdot d\mathbf{r} + \int_b^c \mathbf{B} \cdot d\mathbf{r} + \int_c^d \mathbf{B} \cdot d\mathbf{r} + \int_d^a \mathbf{B} \cdot d\mathbf{r} = \mu_0 I \tag{3.4}$$

The first integral is given by a magnetic field $\mathbf{B}$ times the length from a to b. The second and fourth integral are equal 0, because the angle with the amperian is 90, so

Figure 7 – Amperian loop.



Source: Author (2021).
Caption: The amperian loop "abcd" and size l in a cut view of the interior of the solenoid.

there's no magnetic field generated. The third integral is also 0 when the path isn't crossing any magnetic field, as shown below:

$$\int \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{r} = \mu_0 N I \tag{3.5}$$

The electric current intern is given by product of the electric current by the density of turns n by its length l. Therefore:

$$\oint \mathbf{B} \cdot d\mathbf{r} = Bl + 0 + 0 + 0 = \mu_0 I(nl) \tag{3.6}$$
$$Bl = \mu_0 I(nl) \tag{3.7}$$
$$B = \mu_0 I n \tag{3.8}$$

Knowing that the reference point is inside the solenoid and by the right-hand rule, we can calculate the magnetic field $\mathbf{B}_{in}$:

$$\mathbf{B}_{in} = \mu_0 n I \hat{z} \tag{3.9}$$

Remembering that outside there isn't any electric current, $I_{out} = 0$, therefore:

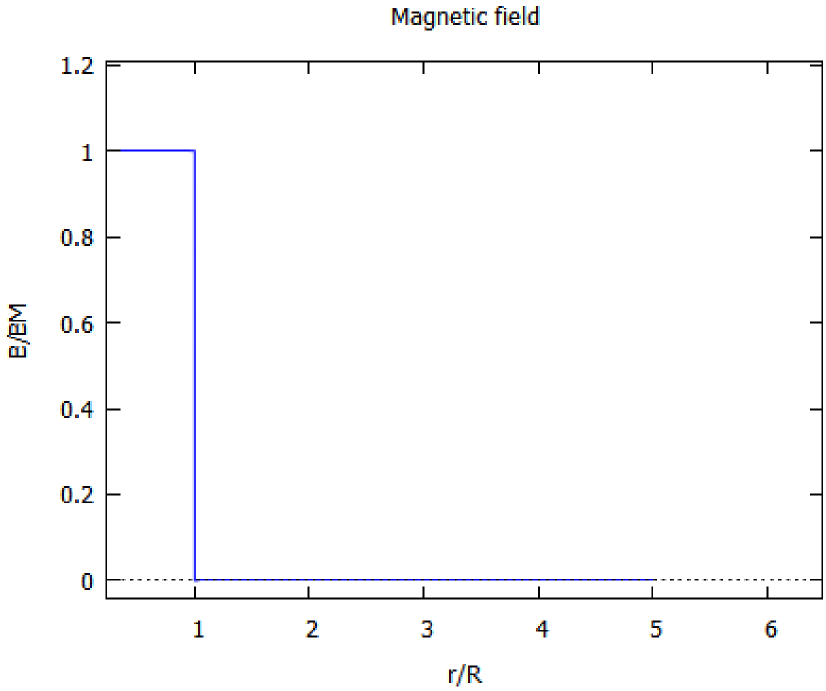
$$\mathbf{B}_{out} = 0 \tag{3.10}$$

Non dimentionally, we can graph the external magnetic field using B_M as non-dimentional constant as seen calculated in 3.3 and graphically, in the Figure 8.

$$\frac{\mathbf{B}_{out}}{B_M} = 0 \tag{3.11}$$
$$\mathbf{B}_M \equiv \mu_0 n I \tag{3.12}$$

In the Figure 8 it is possible to observe that the non dimensional intern magnetic field exists only inside the radius of the solenoid and when we cross out of it, it goes immediately to zero.

Figure 8 – Non dimensional magnetic field in Maxwell.



Source: Author (2021).
Caption: Internal and external magnetic field on a solenoid.

3.4 VECTOR POTENTIAL A

Possessing the value of the magnetic field, we can follow our script for calculating the vector potential. As there are no isolated magnetic charges; its definition comes from the vector calculation, which states that the deviation from the rotational of any function vector is null. Thus, by rotating the potential vector, we can find the magnetic field, as shown next.

3.4.1 Internal Potential

For intern vector potential, the system presents a magnetic vector potential **A** along the magnetic field **B**, and it can be defined by the equation 3.13 below:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.13}$$

solving this using equation 3.9 shows us how **A** should be presented in cylindrical coordinates, $\mathbf{A} = (A_r, A_\theta, A_z)$, then:

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\partial_\phi A_z - \partial_z\left(rA_\phi\right)\right]\partial_z A_r - \partial_r A_z \frac{1}{r}\left[\partial_r\left(rA_\phi\right) - \partial_\phi A_r\right] = \mu_0 n I \hat{z} \tag{3.14}$$

Equating the results, the system is now as described below and can infer some values, so component $A_z = 0$, in line with:

$$\begin{aligned} \left[\frac{1}{r}\partial_\phi A_z - \partial_z\left(rA_\phi\right)\right] &= 0 \quad (3.15)\\ \partial_z A_r - \partial_r A_z &= 0 \quad (3.16)\\ \frac{1}{r}\left[\partial_r\left(rA_\phi\right) - \partial_\phi A_r\right] &= \mu_0 nI \quad (3.17) \end{aligned}$$

By the symmetry, that $\mathbf{A}(r)$ only , then:

$$\begin{aligned} \partial_r A_z &= 0 \quad (3.18)\\ \frac{1}{r}\left[\partial_r\left(rA_\phi\right)\right] &= \mu_0 nI \end{aligned}$$

The system of equations 4.40 above can only have those results if its derivatives are some kind of function that is r dependent. Resulting in:

$$\frac{1}{r}\partial_r\left(rA_\phi\right) = \mu_0 nI \quad (3.19)$$

$$\begin{aligned} A_\phi &= \frac{c\left(r\right)}{r} = \frac{\mu_0 nIr^2 + k}{2r}\\ A_z &= d \quad (3.20) \end{aligned}$$

where $c\left(r\right)$,$d\left(r\right)$ and k are a constants.

Using Coulomb's gauge to simplify the equations:

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\partial_r\left(rA_r\right) + \frac{1}{r}\partial_\phi A_\phi + \partial_z A_z = 0 \quad (3.21)$$

in 3.20 and using y as an independent constant.

$$\partial_r\left(rA_r\right) = 0 \rightarrow A_r = \frac{y}{r} \quad (3.22)$$

Assembling these results, the magnetic vector potential can now be written as:

$$\mathbf{A}\left(r\right) = \frac{y}{r}\hat{r} + \left(\mu_0 nI\frac{r}{2} + \frac{k}{r}\right)\hat{\phi} \quad (3.23)$$

But this form can be simplified if we take $y = 0 = k$ since both are undefined constants, so its final form is given by:

$$\mathbf{A}_{in}\left(r\right) = \mu_0 nI\frac{r}{2}\hat{\phi} \quad (3.24)$$

Graphically, the internal vector potential can be draw as seen in Figure 9.

In the Figure 9, we can see how the vector potential increases as further from the solenoid center, in direction to the solenoid radius.

3.4.2 External Potential

Outside the solenoid, the magnetic field measured is zero, as shown by the approximation of the radius r to distance R very far from the point considered earlier. However, this doesn't imply that the magnetic vector potential is also zero. It's known that the magnetic flux can be calculated applying the Stoke's Theorem in the magnetic field, then:

$$\Phi_{\mathbf{B}} = \int_s \mathbf{B} \cdot d\mathbf{r} = \int_C \mathbf{A} \cdot d\mathbf{l} \tag{3.25}$$

Considerating:

$$d\mathbf{l} = dl\hat{\phi} \tag{3.26}$$

$$\mathbf{A} \cdot d\mathbf{l} = A_\phi(r)dl \tag{3.27}$$

$$dl = sd\phi \tag{3.28}$$

Solving the right side of the equation:

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = A_\phi(r)rd\phi = A_\phi(r)r2\pi \tag{3.29}$$

$$\begin{aligned} \Phi_{\mathbf{B}} &= \mu_0 I\pi R^2 \\ &= \mathbf{A}(r).2\pi r \end{aligned} \tag{3.30}$$

Equating 3.29 and 3.30, A_ϕ is given:

$$A_\phi(r)2\pi r = \mu_0 I\pi R^2 \tag{3.31}$$

$$A_\phi(r) = \frac{\mu_0 I R^2}{2r} \tag{3.32}$$

We define $A_z = 0 = A_r$ and calculate the other components, achieving the final form of $\mathbf{A}$:

$$\mathbf{A}_{out}(r) = \frac{\mu_0 n I R^2}{2r}\hat{\phi} \tag{3.33}$$

Where the vector potential in Maxwell is:

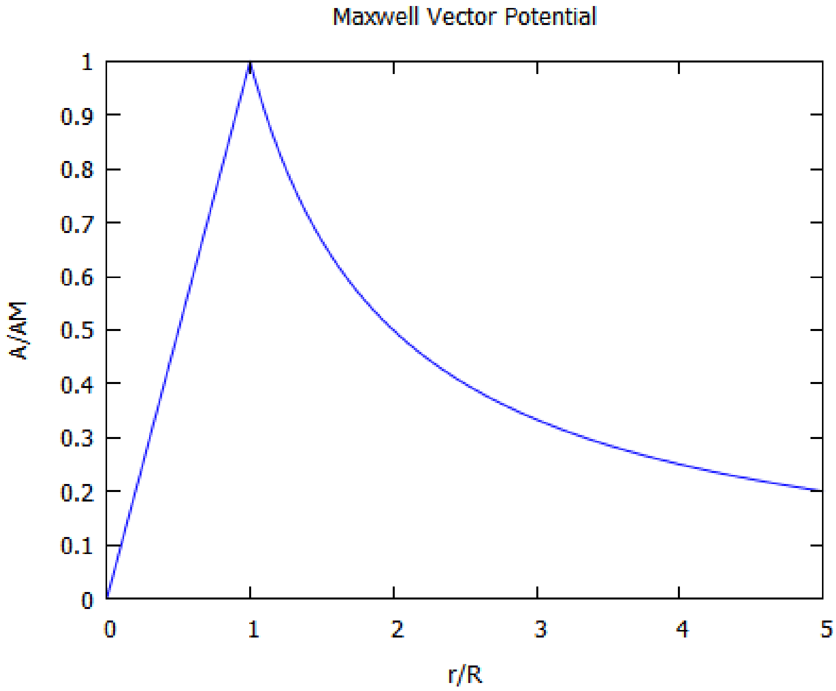
$$A_M(r) = \frac{\mu_0 I R}{2} \tag{3.34}$$

Graphically, the non dimensional external potential can be seen in the Graphic 9 and its possible to see that it gently decreases by $\frac{1}{r}$ to zero while moving away from the radius of the solenoid, since the derivative of $\mathbf{A}$ is discontinuous, therefore it is the definition of the magnetic field.

Analysing the whole system, graphically it behaves as shown in Figure 9 where the potential grows quickly when the distance considered is smaller than the radius and once it begins to decay the distance from the radius is getting bigger, tending to the infinity, where it should be equal to zero.

Figure 9 presents the complete vector potential, its behavior inside and out the solenoid, and the lack of continuity when the radius is reached.

Figure 9 – Non dimensional vector potential in Maxwell.



Source: Author (2021).
Caption: External and internal vector potential on a solenoid.

3.5 SCHRÖDINGER'S EQUATION

In classical Electrodynamics potentials V and $\mathbf{A}$ are not measured, but we do measure the magnetic andd electric fields and those potentials are then calculated with:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{3.35}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3.36}$$

Maxwell equation's and the Lorentz force rule do not include the potentials for those are only theoretical references, which can be changed without altering the field, likewise a gauge transformation applied on those potentials.

$$V \rightarrow V' = V - \frac{\partial \Lambda}{\partial t} \tag{3.37}$$

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \Delta\Lambda \tag{3.38}$$

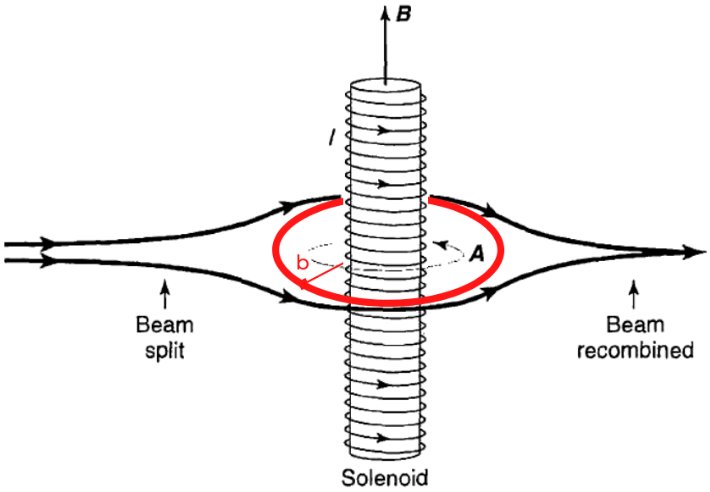
Λ being a function in time and space. In Quantum Mechanics, potentials are meaningful, shown in the hamiltonian:

$$H = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + qV \tag{3.39}$$

but this also doesn't change with gauge transformations.

In the past, it was a common belief that those potentials would not interfere in the fields in places where **B** and **E** are null. Until 1959, Aharonov and Bohm did an experiment, as seen in Figure 10 that showed was the effect of the potential vector can affect the quantum behavior of a charged particle, even though it moves in a q region where the field is null.

Figure 10 – Schematic of the AB experiment.



Source: Author (2021).
Caption: Path b of the particles outside the solenoid.

Considering a particle in a circular movement with radius b. In the axis, there's a solenoid with radius $r < R$ charged with a current I. If the solenoid is extremely long, the magnetic field inside is uniform and the magnetic field outside is zero, although the potential vector it's not zero, as calculated before.

$$\mathbf{A}_{out} = \frac{\Phi}{2\pi r}\,\hat{\phi} \tag{3.40}$$

$$\nabla = \frac{\hat{\phi}}{r}\frac{d}{d\phi} \tag{3.41}$$

and time independent Schrödinger's equation:

$$\frac{1}{2m}\left[\frac{-\hbar^2}{r^2}\frac{d^2\psi}{d\phi^2} + q^2\mathbf{A}^2\psi + \frac{2i\hbar q}{r}\mathbf{A}\cdot\hat{\phi}\frac{d\psi}{d\phi}\right] = E\psi \tag{3.42}$$

Using 3.40

$$\frac{-\hbar^2}{r^2}\frac{d^2\psi}{d\phi^2} + q^2\left(\frac{\Phi}{2\pi r}\right)^2\psi + \frac{2i\hbar q}{\pi r^2}\left(\frac{\Phi}{2\pi r}\right)\frac{d\psi}{d\phi} = 2mE\psi \tag{3.43}$$

Multiplying by $\frac{-r^2}{\hbar^2}$ then:

$$\frac{d^2\psi}{d\phi^2} - \frac{r^2}{\hbar^2}q^2\left(\frac{\Phi}{2\pi r}\right)^2\psi - \frac{r^2}{\hbar^2}\frac{2i\hbar q}{\pi r^2}\left(\frac{\Phi}{2\pi r}\right)\frac{d\psi}{d\phi} = \frac{-r^2}{\hbar^2}2mE\psi \tag{3.44}$$

being β and ε defined as:

$$\beta \equiv \frac{q\Phi}{2\pi\hbar} \tag{3.45}$$

$$\varepsilon \equiv \frac{2mr^2E}{\hbar^2} - \beta^2 \tag{3.46}$$

so we have the differencial equation:

$$\frac{d^2\psi}{d\phi^2} - 2i\beta\frac{d\psi}{d\phi} + \varepsilon\psi = 0 \tag{3.47}$$

with solution as $\psi = \psi_0 e^{\zeta\phi}$ where:

$$\frac{d\psi}{d\phi} = \psi_0 e^{\zeta\phi}\varsigma \tag{3.48}$$

$$\frac{d^2\psi}{d\phi^2} = \psi_0 e^{\zeta\phi}\varsigma^2$$

Using 3.48 in 3.47 :

$$\varsigma^2 - 2i\beta\varsigma + \varepsilon = 0 \tag{3.49}$$

where

$$\lambda = \beta \pm \sqrt{\beta^2 + \varepsilon} \tag{3.50}$$

or

$$\lambda = \beta \pm \frac{b}{\hbar}\sqrt{2mE} \tag{3.51}$$

By the boundary conditions fixed, ψ should be continuous after one turn around the solenoid, so:

$$\psi_{(\phi)} = \psi_{(\phi+2\pi)} \tag{3.52}$$

$$\psi_0 e^{i\lambda\phi} = \psi_0 e^{i\lambda(\phi+2\pi)} \tag{3.53}$$

By Euler's equation:

$$1 = e^{i2\pi} \tag{3.54}$$

Showing that

$$\lambda = n \tag{3.55}$$

$$n = \{0, \pm 1, \pm 2, \pm 3...\}$$

Replacing 3.55 in 3.51

$$\beta \pm \frac{r}{\hbar}\sqrt{2mE} = n \tag{3.56}$$

therefore, the energy is:

$$E_n = \frac{\hbar^2}{2mr^2}\left(n - \frac{q\Phi}{2\pi\hbar}\right)^2 \tag{3.57}$$

3.6 AHARONOV-BOHM PHASE

In 1959, Aharonov-Bohm proposed that the phase should be equal to the equation 3.60 when studying the AB experiment [1]. In this section, we will calculate the AB phase in Maxwell, using the ansatz 3.60 that Aharonov and Bohm proposed from the Berry phase 3.58:

$$g(\mathbf{r}) \equiv \frac{q}{\hbar}\int_{\mathcal{O}}^{r} \mathbf{A} \cdot d\mathbf{r}' \tag{3.58}$$

Replacing the potential **A** in Schrödinger's equation:

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + V\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t} \tag{3.59}$$

the solution that should be similar to:

$$\Psi = e^{ig}\Psi' \tag{3.60}$$

Notice that:

$$\begin{aligned} \nabla\Psi &= ie^{ig}\Psi' + e^{ig}\nabla\Psi' && (3.61)\\ \frac{\hbar}{i}\nabla\Psi &= q\mathbf{A}e^{ig}\Psi' + \frac{\hbar}{i}e^{ig}\nabla\Psi' && (3.62)\\ \left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\Psi &= \frac{\hbar}{i}e^{ig}\nabla\Psi' && (3.63) \end{aligned}$$

Solving separately, the right side first:

$$\begin{aligned} \left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\Psi &= \left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)e^{ig}\nabla\Psi' && (3.64)\\ &= -\hbar^2\nabla\cdot\left(e^{ig}\nabla\Psi'\right) - \frac{q\hbar}{i}e^{ig}\mathbf{A}\cdot\nabla\Psi' && (3.65)\\ &= -i\hbar q e^{ig}\mathbf{A}\cdot\nabla\Psi' - \hbar^2 e^{ig}\nabla^2\Psi' + iq\hbar e^{ig}\mathbf{A}\cdot\nabla\Psi' && (3.66)\\ &= -\hbar^2 e^{ig}\nabla^2\Psi' && (3.67) \end{aligned}$$

then the left side:

$$\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\Psi = \left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)\left(\frac{\hbar}{i}\nabla\Psi - q\mathbf{A}\Psi\right) \tag{3.68}$$

$$= -\hbar^2\nabla^2\Psi\frac{\hbar q}{i}\nabla\cdot(\mathbf{A}\Psi) - \frac{\hbar q}{i}\mathbf{A}\nabla\Psi + q^2\mathbf{A}^2\Psi \tag{3.69}$$

$$= -\hbar^2\nabla^2\Psi\frac{\hbar q}{i}\left(\Psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\Psi\right) - \frac{\hbar q}{i}\mathbf{A}\cdot\nabla\Psi + \tag{3.70}$$

$$+q^2\mathbf{A}^2\Psi \tag{3.71}$$

$$= -\hbar^2\nabla^2\Psi - 2\frac{\hbar q}{i}\mathbf{A}\cdot\nabla\Psi + q^2\mathbf{A}^2\Psi\frac{\hbar q}{i}\Psi\nabla\cdot\mathbf{A} \tag{3.72}$$

$$= \left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2\Psi \tag{3.73}$$

Since $\nabla\cdot\mathbf{A}$ is null, the equation is correct for the both sides. Then, replacing 3.60 and 3.62 into 3.59:

$$\frac{1}{2m} - \hbar^2\nabla^2\Psi' + Ve^{ig}\Psi' = -\frac{\hbar}{i}\frac{\partial(e^{ig}\Psi')}{\partial t} \tag{3.74}$$

Replacing $\mathbf{A}_{out}$ in 3.58:

$$g(\mathbf{r}) \equiv \frac{q}{\hbar}\int_0^{2\pi}\frac{\Phi}{2\pi b}\hat{\phi}\cdot\hat{\phi}\,(\pm b d\phi) \tag{3.75}$$

$$g_{\pm} = \pm\frac{q}{\hbar}\frac{\Phi}{2\pi}\int_0^{2\pi} d\phi \tag{3.76}$$

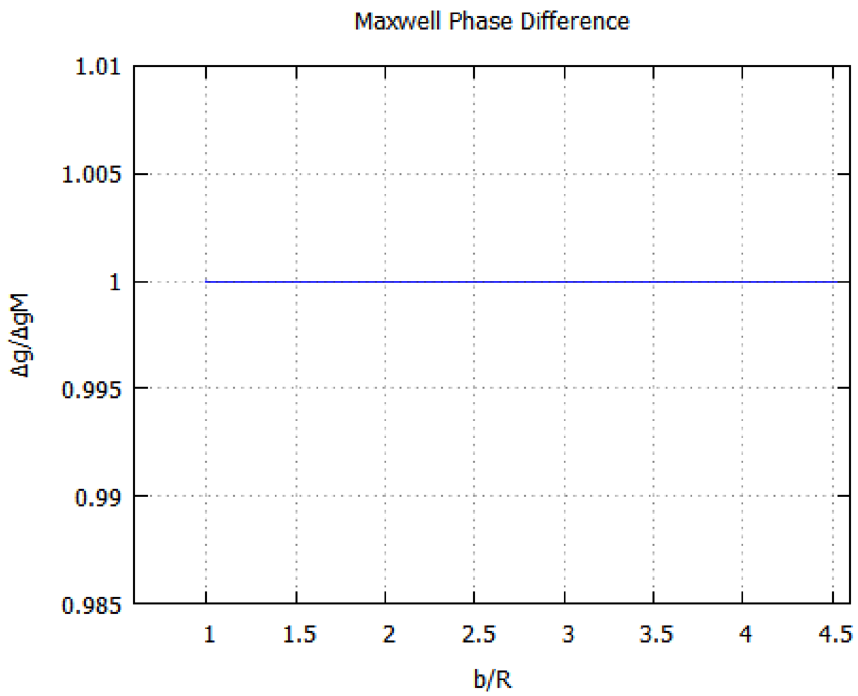
$$g_{\pm} = \pm\frac{q\Phi}{2\hbar} \tag{3.77}$$

$$\Delta g_M = \frac{q\Phi}{\hbar} \tag{3.78}$$

This result can also be graphically demonstrated in a non-dimensional representation of the phase difference in Maxwell, that is linear and continuous, once it does not alternates, as seen in Figure 11.

In the next section, the same analysis will be done for the case of Podolsky's electrodynamics, to study possible differences between the theories, which, if any, can be used as an experimental test of Podolsky's theory.

Figure 11 – Non dimensional AB phase in Maxwell.



Source: Author (2021).
Caption: The constant AB phase in Maxwell.

4 AHANOROV-BOHM EFFECT IN PODOLSKY ELECTRODYNAMICS

4.1 MAGNETIC FIELD IN A LONG SOLENOID

The solenoid as we are studying so far still has the same physical properties and so is the experimental apparatus. So in the coil interior, the magnetic field is being generated by the electric current flowing in it and outside, the magnetic field now might have some small values, that decrease fast as it goes further from its center. This calculation of how much small and closest from zero magnetic fields comes from Podolsky corrections. So the vector potential, in this case, is still valid and somehow improved, since in Podolsky electrodynamics all the equations gain variations that turn the experimental results more accurate. In the sections presented next the script for the calculation is the same we did for Maxwell: calculate the magnetic field inside and out of the solenoid, then calculation the vector potential, the wave equation, and finally the phase shift.

4.2 AMPÈRE-PODOLSKY'S LAW

Once again, to calculate the magnetic field in all regions the system, now in Podolsky electrodynamics, we can stat from Ampère's law equation 4.1:

$$\left(1 + a^2\Box\right)\left(\nabla \times \vec{B} - \frac{\partial \mathbf{E}}{\partial t}\right) = \mathbf{J} \tag{4.1}$$

The electric field is time-invariant, as is the magnetic field:

$$\frac{\partial \mathbf{E}}{\partial t} = 0 \tag{4.2}$$

$$\mathbf{B} = \mathbf{B}\left(\mathbf{x}\right) \tag{4.3}$$

Thus, the second derivative is non-existent, remaining only the Laplacian:

$$\left[1 + a^2\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\left(\nabla \times \mathbf{B}\right) = \mathbf{J} \tag{4.4}$$

$$\left(\nabla \times \mathbf{B}\right) + a^2\left[\frac{\partial^2}{\partial t^2}\left(\nabla \times \mathbf{B}\right) - \nabla^2\left(\nabla \times \mathbf{B}\right)\right] = \mathbf{J} \tag{4.5}$$

$$\left(\nabla \times \mathbf{B}\right) - a^2\nabla^2\left(\nabla \times \mathbf{B}\right) = \mathbf{J} \tag{4.6}$$

We know that in Maxwell's electromagnetism $a = 0$, which reduces the equation for Ampère's Law in the form:

$$\nabla \times \mathbf{B} = \mathbf{J} \tag{4.7}$$

Integrating across the S surface:

$$\int_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} dS = \int_S \mathbf{J} \cdot \mathbf{n} dS \tag{4.8}$$

By Stokes' theorem, we can match the left term with the integral in a boundary C of the field vector along path l, while the right term is the current definition.

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{4.9}$$

Likewise, in Podolsky a cannot assume null values, $a \neq 0$, changing the equation to:

$$(\nabla \times \mathbf{B}) - a^2 \nabla^2 (\nabla \times \mathbf{B}) = \mathbf{J} \tag{4.10}$$

$$(\nabla \times \mathbf{B}) - a^2 \nabla \times \left(\nabla^2 \mathbf{B}\right) = \mathbf{J} \tag{4.11}$$

$$\nabla \times \left[\mathbf{B} - a^2 \nabla^2 \mathbf{B}\right] = \mathbf{J} \tag{4.12}$$

We then define, $\mathbf{B}'$:

$$\mathbf{B}' \equiv \left[\mathbf{B} - a^2 \nabla^2 \mathbf{B}\right] \tag{4.13}$$

So,

$$\nabla \times \mathbf{B}' = \mathbf{J} \tag{4.14}$$

Integrating both sides and in a manner similar to that calculated for $[\mathbf{B} - a^2 \nabla^2 \mathbf{B}]$ Ampere's Law:

$$\oint_C \mathbf{B}' \cdot d\mathbf{l} = \mu_0 I \tag{4.15}$$

Solving for $\mathbf{B}'$ in the solenoid, we have two regions, internal and external that make up the field in its entirety. For $\mathbf{B}'_{out}$, we equal 4.13 to the external field:

$$\mathbf{B}_{out} - a^2 \nabla^2 \mathbf{B}_{out} = 0 \tag{4.16}$$

Multiplying by$-\frac{1}{a^2}$:

$$\nabla^2 \mathbf{B}_{out} - \frac{1}{a^2} \mathbf{B}_{out} = 0 \tag{4.17}$$

For the internal field $\mathbf{B}'_{in}$ by equating 4.13 to the internal field:

$$\mathbf{B}_{in} - a^2 \nabla^2 \mathbf{B}_{in} = \mu_0 n I \hat{z} \tag{4.18}$$

Again, multiplying by $-\frac{1}{a^2}$:

$$\nabla^2 \mathbf{B}_{int} - \frac{1}{a^2}\mathbf{B}_{int} = -\frac{\mu_0 n I}{a^2}\hat{z} \tag{4.19}$$

$$= \left(\nabla^2 B^s_{int} - \frac{1}{a^2}{}^s_{int}\right)\hat{s} + \left(\nabla^2 B^\phi_{int} - \frac{1}{a^2}B^\phi_{int}\right)\hat{\phi} + \left(\nabla^2 B^z_{int} - \frac{1}{a^2}B^z_{int}\right)\hat{z} \tag{4.20}$$

where the components in $\hat{s}$ and $\hat{\phi}$ by the symmetry of the problem are equal to zero, leaving only the component in $\hat{z}$.

4.2.1 External Field

Outside the solenoid, from the Laplacian in cylindrical coordinates, we can rewrite 4.16 as:

$$\frac{\partial^2 B_z}{\partial r^2} + \frac{1}{r}\frac{\partial B_z}{\partial r} - \frac{1}{a^2}B_z = 0 \tag{4.21}$$

Multiplying by r^2 and replacing $m^2 = \frac{1}{a^2}$ we have:

$$r^2\frac{\partial^2 B_z}{\partial r^2} + r\frac{\partial B_z}{\partial r} - m^2 r^2 B_z = 0 \tag{4.22}$$

which has a format similar to the modified Bessel equation 4.23, when $\alpha = 0$:

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} - (x^2 - \alpha^2)y = 0 \tag{4.23}$$

We know that $x = mr$, as a chain's rule, and then:

$$x^2\frac{\partial^2 B_z}{\partial x^2} + x\frac{\partial B_z}{\partial x} - x^2 B_z = 0 \tag{4.24}$$

with known solution for all x belonging to the real numbers given by the modified Bessel functions. The modified Bessel functions are used when the argument in the Bessel functions are imaginary [21] and the solutions can be interpreted with the graphics shown in the Figure 12. Therefore, the solution can be written as:

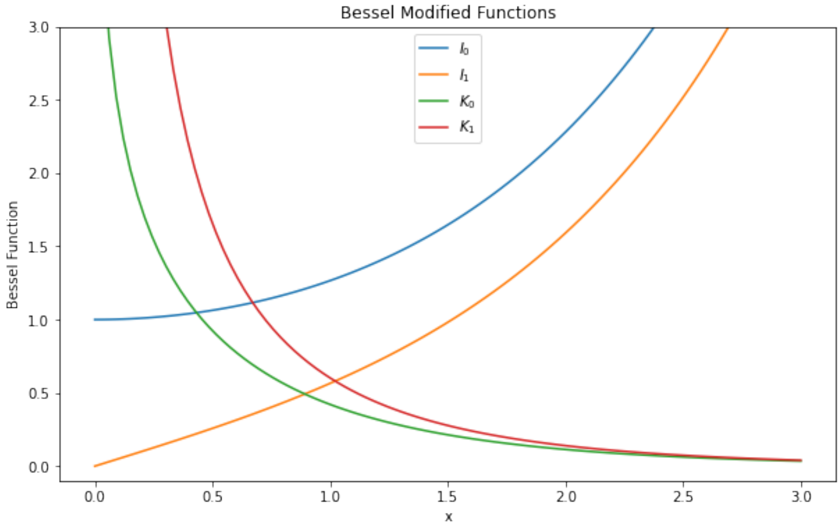
$$B_z(x) = CI_0(x) + DK_0(x) \tag{4.25}$$

$$B_z(x) = DK_0(x) \tag{4.26}$$

where, with the graphic analysis, the solution presented can only have physical significance when $x = 0$, in the interior of the solenoid. So we can assume that $C = 0$, when x tends to infinity, i.e. the solution can only depends on the second term.

Analysing the graphic solution for the external magnetic field in the Figure 13, we can notice that the greater the ω the faster the solution approximates from the Mawxell solution for the external field. Recalling that $\omega = mR = \frac{R}{a}$, so the smaller the Podolsky's constant, greater the ω parameter.

Figure 12 – Modified Bessel functions: $I_0(x), I_1(x), K_0(x)$ and $K_1(x)$ graphics.



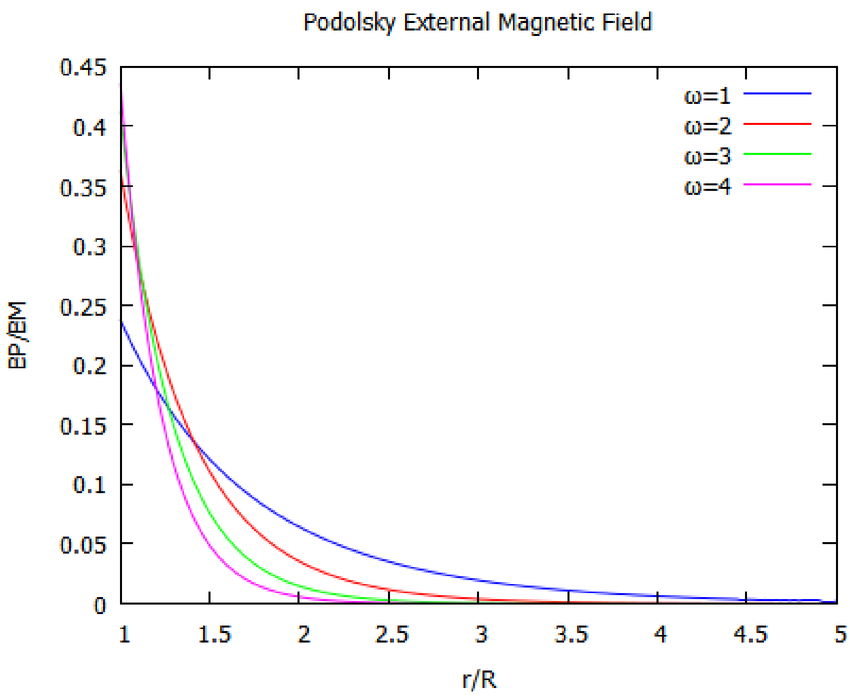
Source: Author (2021).
Caption: Modified Bessel functions.

4.2.2 Internal Field

Considering now its interior, from the Laplacian in cylindrical coordinates, we can rewrite 4.18 as:

$$\frac{\partial^2 B_z}{\partial r^2} + \frac{1}{r}\frac{\partial B_z}{\partial r} - \frac{1}{a^2} B_z = -\frac{\mu_0 n I}{a^2} \tag{4.27}$$

Figure 13 – Non dimensional external field in Podolsky.



Source: Author (2021).
Caption: External magnetic field in Podolsky with ω values from 1 to 4.

Using as a homogeneous solution $y(x) = FI_0(x) + GK_0(x)$ adding to a particular solution.Multiplying by r^2 and replacing $m^2 = \frac{1}{a^2}$, and knowing that $x = mr$:

$$x^2 \frac{\partial^2 B_z}{\partial x^2} + x \frac{\partial B_z}{\partial x} - x^2 B_z = -x^2 \mu_0 n I \tag{4.28}$$

The particular solution is $B_z^p = \mu_0 n I$. Thus, the general solution is::

$$B_z(x) = FI_0(x) + GK_0(x) + B_z^p \tag{4.29}$$

From the continuity, the constant $G = 0$, so the solution is:

$$B_z(x) = FI_0(x) + B_z^p \tag{4.30}$$

From the symmetry of the problem, we must match both in the radius R of the coil, the limit where the field is in tabs the regions, like this:

$$DK_0(x) = FI_0(x) + B_z^p \tag{4.31}$$

$$DK_0(mR) = FI_0(mR) + \mu_0 n I \tag{4.32}$$

$$D = \frac{FI_0(mR) + \mu_0 n I}{K_0(mR)} \tag{4.33}$$

The constant F can be fixed by the continuity boundary condition, explained further below, where $F = \frac{I_0(x)+B_z^p}{DK_0(x}$.

When analysing non dimentionaly the Podolsky intern magnetic field, the non dimensional quantities are calculated and show next:

$$\mathbf{B}_{int}^P = (0,0, -mR\mu_0 n I K_1(mR) I_0(mr) + \mu_0 n I) \tag{4.34}$$

$$B_{int}^P = -mR\mu_0 n I K_1(mR) I_0(mr) + \mu_0 n I \tag{4.35}$$

$$B_M \equiv \mu_0 n I \tag{4.36}$$

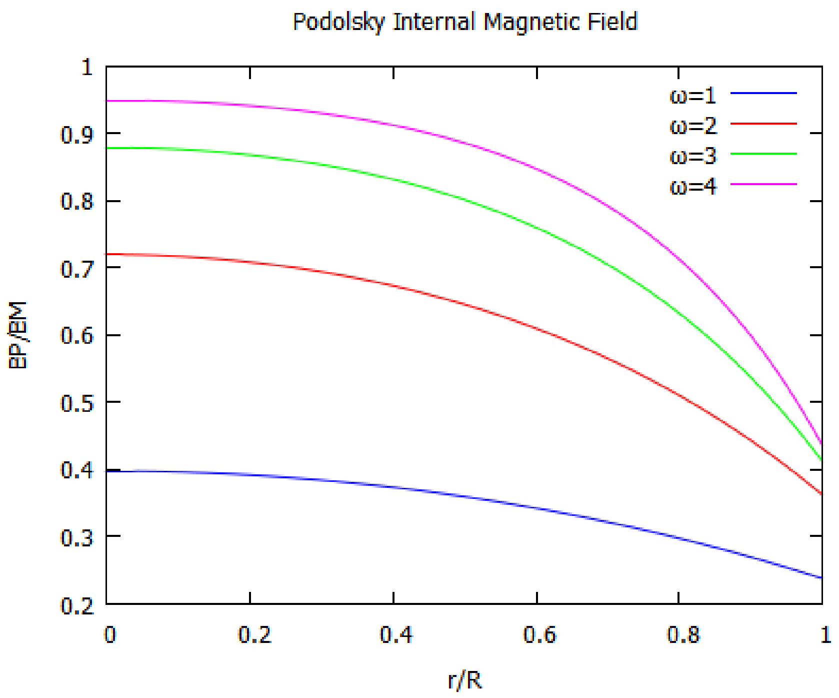
$$B_{int}^P = -mRB_M K_1(mR) I_0(mr) + B_M \tag{4.37}$$

defining $\omega \equiv mR$.

In Figure 14, the different ω represent the values that the Podolsky parameter can assume and how massive the studied system is, being the quantities from 1 to 4 merely demonstrative that the greater the ω, the closest from Maxwell intern magnetic field it will be.

Joining the graphics for internal and external in Figure 15 we realize that the ω pattern observed previously repeats itself, so the higher the ω value, the Maxwell expected result will be closer.

Figure 14 – Non dimensional internal magnetic field in Podolsky



Source: Author (2021).
Caption: Internal magnetic field with ω from 1 to 4.

4.3 VECTOR POTENTIAL A

Once the magnetic field for Podolsky is known, we can find the vector potential. In Maxwell's theory, the vector potential was only a mathematical artifact, in Podolsky, it can be proven a real quantity, even though nowadays, only theoretical still with few experimental results. Still, can be calculated by rotating the potential vector and returning the magnetic field, as shown in this section.

4.3.1 Internal Potential

Inside the solenoid, from Maxwell electrodynamics:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.38}$$

using 3.9, we notice that $\mathbf{A}$ must be in cylindrical coordinates, $\mathbf{A} = (A_r, A_\theta, A_z)$, then:

$$\nabla \times \mathbf{A} = \left[\frac{1}{r}\partial_\phi A_z - \partial_z \left(rA_\phi\right)\right] \partial_z A_r - \partial_r A_z \frac{1}{r}\left[\partial_r \left(rA_\phi\right) - \partial_\phi A_r\right] (4.39)$$

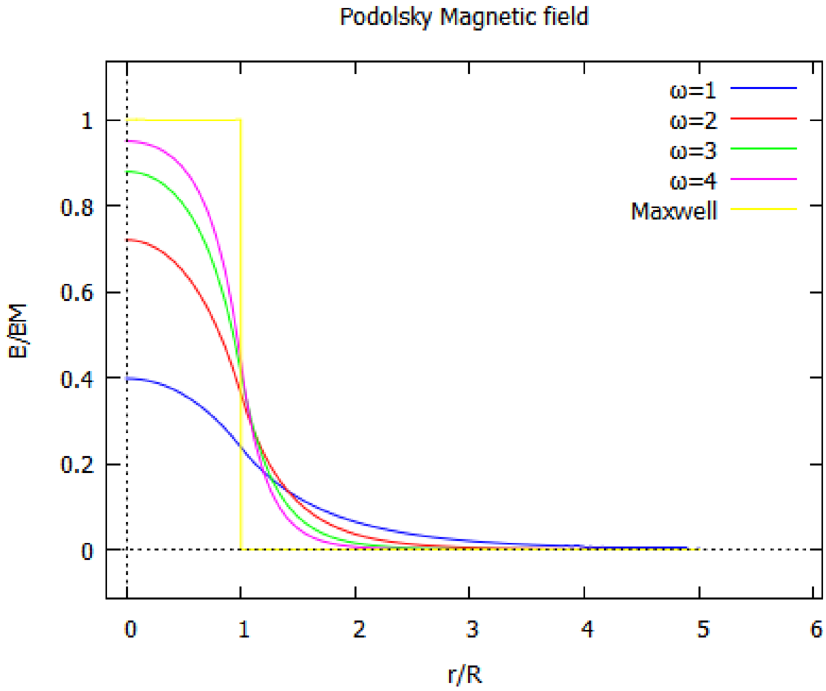
=FI$_0(mr) + B_z^p$(4.39)Equating the results, we can define that $A_z = 0$ and $x = mr$:

$$\left[\frac{1}{r}\partial_\phi A_z - \partial_z \left(rA_\phi\right)\right] = 0$$

$$\partial_z A_r - \partial_r A_z = 0$$

$$\frac{1}{r}\left[\partial_r \left(rA_\phi\right) - \partial_\phi A_r\right] = FI_0(mr) + B_z^p$$

Figure 15 – Non dimensional magnetic field in Podolsky



Source: Author (2021).
Caption: External and internal magnetic field with ω from 1 to 4.

If the potential depends only on $\mathbf{A}(r)$:

$$\partial_r A_z = 0 \tag{4.40}$$
$$\frac{1}{r}\left[\partial_r\left(rA_\phi\right)\right] = FI_0(mr) + B_z^p$$

Since the derivative in r of A_z is zero, we know that $A_z = \kappa$. Solving the second equation, we get:

$$\partial_r\left(rA_\phi\right) = rFI_0(mr) + rB_z^p \tag{4.41}$$
$$rA_\phi = \int rFI_0(mr)dr + \frac{r^2}{2}B_z^p + j \tag{4.42}$$
$$A_\phi = \frac{\int rFI_0(mr)dr}{r} + \frac{rB_z^p}{2} + \frac{j}{r} \tag{4.43}$$

Solving the integral separately:

$$F\int rI_0(mr)dr \tag{4.44}$$

Knowing that $I_0 = I_\nu(x)$ when $\nu = 0$:

$$I_\nu(x) = \sum_{s=0}^{\infty}\frac{1}{s!(s+\nu)!}\left(\frac{x}{2}\right)^{2s+\nu} \tag{4.45}$$
$$I_0(mr) = \sum_{i=0}^{\infty}\frac{1}{(i)!(i)!}\left(\frac{mr}{2}\right)^{2i} \tag{4.46}$$

Solving the integral, separately:

$$\int rI_0(mr)dr = \sum_{i=0}^{\infty}\left[\frac{1}{(i)!(i)!}\left(\frac{m}{2}\right)^{2i}\right]\int r^{2i+1}dr \tag{4.47}$$

$$= \sum_{i=0}^{\infty}\left[\frac{1}{(i)!(i)!}\left(\frac{m}{2}\right)^{2i}\right]\frac{r^{2i+2}}{2i+2} \tag{4.48}$$

$$= \frac{r}{m}\sum_{i=0}^{\infty}\frac{1}{(i)!}\frac{1}{(i+1)!}\left(\frac{m}{2}r\right)^{2i+1} \tag{4.49}$$

$$= \frac{r}{m}I_1(mr) \tag{4.50}$$

Replacing in 4.44:

$$F\int rI_0(mr)dr = F\frac{r}{m}I_1(mr) \tag{4.51}$$

Once more, replacing, in:

$$A_\phi = \frac{FI_1(mr)}{m} + \frac{rB_z^p}{2} + \frac{j}{r} \tag{4.52}$$

When r=0, A_ϕ goes to infinity and then j has to be zero, so:

$$A_\phi = \frac{FI_1(mr)}{m} + \frac{rB_z^p}{2} \tag{4.53}$$

From proprieties in I_ν:

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x}I_\nu(x) \tag{4.54}$$

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I'_\nu(x) \tag{4.55}$$

Solving, we can define $U_r = \frac{1}{r}\frac{\partial}{\partial r}(rA_r)$

$$U_r - a^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(U_r)\right)\right] = 0 \tag{4.56}$$

Multiplying the entire equation by $-\frac{1}{a^2}$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(U_r)\right) - \frac{1}{a^2}U_r = 0 \tag{4.57}$$

Defining $m^2 = \frac{1}{a^2}$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(U_r)\right) - m^2U_r = 0 \tag{4.58}$$

$$\frac{\partial^2}{\partial r^2}U_r + \frac{1}{r}\frac{\partial U_r}{\partial r} - m^2U_r = 0 \tag{4.59}$$

Multiplying the entire equation by r^2:

$$r^2\frac{\partial^2}{\partial r^2}U_r + r\frac{\partial U_r}{\partial r} - r^2m^2U_r = 0 \tag{4.60}$$

Defining $x = mr$:

$$x^2\frac{\partial^2}{\partial x^2}U_r + x\frac{\partial U_r}{\partial x} - x^2U_r = 0 \tag{4.61}$$

Which has a similar format to the modified Bessel equation when $\alpha = 0$, so U_r has a known solution for α in the real numbers:

$$U_r\,(x) = MI_0(x) + NK_0(x) \tag{4.62}$$
$$U_r\,(mr) = MI_0(mr) + NK_0(mr) \tag{4.63}$$

Replacing in 4.3.1:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) = MI_0(mr) + NK_0(mr) \tag{4.64}$$
$$rA_r = M\int rI_0(mr)dr + N\int rK_0(mr)dr + w \tag{4.65}$$

Solving the integral:

$$\int rK_0(mr)dr = \frac{-rK_1(mr)}{m} + e \tag{4.66}$$

replacing the integral, we then have that A_r:

$$A_r = \frac{M}{m}I_1(mr) - \frac{N}{m}K_1(mr) + \frac{w}{r} \tag{4.67}$$

When $r = 0$, K_1 goes to infinity so we can consider $N = 0$ and $w = 0$, resulting in:

$$A_r = \frac{M}{m}I_1(mr) \tag{4.68}$$

Thus, the components of $\mathbf{A}$ in the internal case of the coil are:

$$\mathbf{A}= (A_r,A_\phi,A_z) \tag{4.69}$$
$$\mathbf{A}^P_{int} = \left(\frac{M}{m}I_1(mr),\frac{FI_1(mr)}{m} + \frac{rB^p_z}{2},\kappa\right) \tag{4.70}$$

When calculation the non dimensional internal potential, we can analyse these results better, as following the calculus presented.

$$\mathbf{A}^P_{int}(r)=\left(0,\frac{FI_1(mr)}{m} + \frac{r\mu_0 nI}{2},0\right) \tag{4.71}$$

The constants F and D are then defined in the next equations by the boundary condition, explained in Section 4.4.

$$F = -mR\mu_0 nIK_1(mR) \tag{4.72}$$
$$D = mR\mu_0 nII_1(mR) \tag{4.73}$$

Replacing F in the intern potential vector:

$$A^P_{int}(r)=-\,R\mu_0 nIK_1(mR)I_1(mr) + \frac{r\mu_0 nI}{2} \tag{4.74}$$

Defining and replacing:

$$A^M \equiv \mu_0 n I R = [Tm] \tag{4.75}$$

$$A^P_{int}(r) = -A^M K_1(mR) I_1(mr) + \frac{A^M r}{2R} \tag{4.76}$$

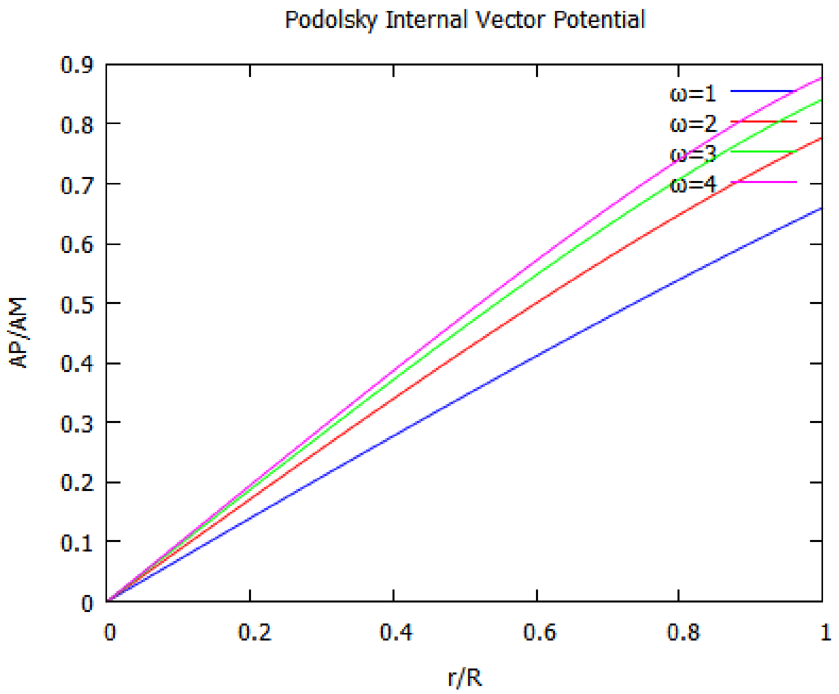
$$\frac{A^P_{int}(r)}{A^M} = -K_1(mR) I_1(mr) + \frac{r}{2R} \tag{4.77}$$

$$\omega \equiv mR \rightarrow m = \frac{\omega}{R} \tag{4.78}$$

$$\chi \equiv \frac{r}{R} \tag{4.79}$$

$$\frac{A^P_{int}(\chi,\omega)}{A^M} = \frac{\chi}{2} - K_1(\omega) I_1(\omega\chi) \tag{4.80}$$

Figure 16 – Non dimensional internal vector potential in Podolsky



Source: Author (2021).
Caption: Internal vector potential with ω from 1 to 4.

4.3.2 External Potential

We can calculate the external vector potential by using the Podolsky solution for the external magnetic field $B_z = DK_0(mr)$ and $\mathbf{B} = \nabla \times \mathbf{A}$, solving in cylindrical coordinates and knowing that $\mathbf{A}$ only depends on r, so:

$$\nabla \times \mathbf{A} = \left(\left[\frac{1}{r}\partial_\phi A_z - \partial_z\left(rA_\phi\right)\right] \quad \partial_z A_r - \partial_r A_z \quad \frac{1}{r}\left[\partial_r\left(rA_\phi\right) - \partial_\phi A_r\right]\right) = DK_0(mr)\hat{z} \tag{4.81}$$

$$\left[\frac{1}{r}\partial_\phi A_z - \partial_z\left(rA_\phi\right)\right] = 0$$
$$\partial_z A_r - \partial_r A_z = 0$$
$$\frac{1}{r}\left[\partial_r\left(rA_\phi\right) - \partial_\phi A_r\right] = DK_0(mr)$$

We get:

$$\partial_r A_z = 0 \tag{4.82}$$

$$\frac{1}{r}\partial_r\left(rA_\phi\right) = DK_0(mr) \tag{4.83}$$

Since the derivative in z is zero, we know that A_z is a contanst γ. From the second equation:

$$\partial_r\left(rA_\phi\right) = rDK_0(mr) \tag{4.84}$$
$$rA_\phi = D\int rK_0(mr)dr + u \tag{4.85}$$

Replacing in 4.85, then r and t are independent, they change the order, integrate in r and try to return the shape with K_0 Solving K_ν:

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x) \tag{4.86}$$
$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x) \tag{4.87}$$

Adding both:

$$K_{\nu-1}(x) = -\frac{\nu}{x}K_\nu(x) - K'_\nu(x) \tag{4.88}$$

Multiplying by x and integrating in x, on both sides:

$$\int xK_{\nu-1}(x)dx = \int\left[-\nu K_\nu(x) - xK'_\nu(x)\right]dx \tag{4.89}$$

For $\nu = 1$:

$$\int xK_0(x)dx = -\int\left[K_1(x) + xK'_1(x)\right]dx \tag{4.90}$$
$$\int xK_0(x)dx = -\int\frac{d}{dx}\left[xK_1(x)\right]dx \tag{4.91}$$
$$\int xK_0(x)dx = -xK_1(x) + c \tag{4.92}$$

Replacing $x = mr$:

$$\int mrK_0(mr)dr = -rK_1(mr) + d \tag{4.93}$$
$$\int rK_0(mr)dr = \frac{-rK_1(mr)}{m} + e \tag{4.94}$$

Replacing:

$$A_\phi = -\frac{DK_1(mr)}{m} + \frac{f}{r} \tag{4.95}$$

From the generalized Coulomb condition from Pimentel and Galvão [22]:

$$\left(1 - a^2\nabla^2\right)\nabla\cdot\mathbf{A} = 0 \tag{4.96}$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) \tag{4.97}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) \tag{4.98}$$

Multiplying by $-\frac{1}{2a}$:

$$\left(1 - a^2\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r})\right)\frac{1}{r}\frac{\partial}{\partial r}(rA_r) = 0 \tag{4.99}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}rA_r)\right] - \frac{1}{a^2}\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) = 0 \tag{4.100}$$

Defining:

$$U_R = \frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) \tag{4.101}$$

Replacing in the previous equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(U_R)\right] - \frac{1}{a^2}U_R = 0 \tag{4.102}$$

Defining $m^2 = \frac{1}{a^2}$:

$$\frac{\partial^2}{\partial r^2}(U_R) + \frac{1}{r}\frac{\partial}{\partial r}(U_R) - m^2U_R = 0 \tag{4.103}$$

Multiplying by r^2 and defining $x = mr$:

$$x^2\frac{\partial^2}{\partial x^2}U_R + x\frac{\partial U_R}{\partial x} - x^2U_R = 0 \tag{4.104}$$

Which has a similar format to the modified Bessel equation when $\alpha = 0$, so U_R has a known solution for U belonging to the reals:

$$U_R\left(mr\right) = PI_0(mr) + QK_0(mr) \tag{4.105}$$

According to the graph, for the outer region of the coil, $I_0 = 0$, so only the last term of the solution prevails:

$$U_R\left(mr\right) = QK_0(mr) \tag{4.106}$$

replacing the value of U_R:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right) = QK_0(mr) \tag{4.107}$$

Multiplying by r and integrating:

$$\int\left(rA_r\right)dr = Q\int rK_0(mr)dr \tag{4.108}$$

where the right integral, previously solved, is:

$$\int rK_0(mr)dr = -\frac{rK_1(mr)}{m} + s \tag{4.109}$$

and then:

$$A_r = -\frac{QK_1(mr)}{m} + \frac{s}{r} \tag{4.110}$$

With that, we have the potential external vector:

$$\mathbf{A}{=}(A_r{,}A_\phi{,}A_z) \tag{4.111}$$

$$\mathbf{A}_{ext} = \left(-\frac{QK_1(mr)}{m} + \frac{s}{r}, -\frac{DK_1(mr)}{m} + \frac{f}{r}{,}\gamma\right) \tag{4.112}$$

In the region where $r = R$, where R is the coil radius, the potential is equal in both regions:

$$\mathbf{A}^P_{int} = \mathbf{A}^P_{ext} \tag{4.113}$$

$$\frac{M}{m}I_1(mR) + \frac{w}{R} = -\frac{QK_1(mR)}{m} + \frac{s}{R} \tag{4.114}$$

$$\frac{FI_1(mR)}{m} + \frac{RB^p_z}{2} + \frac{j}{R} = -\frac{DK_1(mR)}{m} + \frac{f}{R} \tag{4.115}$$

$$\kappa = \gamma \tag{4.116}$$

The magnetic field for Podolsky's case, when m tends to infinity, has to converge to Maxwell's magnetic field (the same is true for the vector potential):

$$\mathbf{B}^M_{ext} = \lim_{m\to\infty} \mathbf{B}^P_{ext} \tag{4.117}$$

$$\mathbf{B}^M_{int} = \lim_{m\to\infty} \mathbf{B}^P_{int} \tag{4.118}$$

$$\mathbf{A}^M_{ext} = \lim_{m\to\infty} \mathbf{A}^P_{ext} \tag{4.119}$$

$$\mathbf{A}^M_{int} = \lim_{m\to\infty} \mathbf{A}^P_{int} \tag{4.120}$$

Remembering that:

$$\mathbf{A}^M_{ext}(r) = \frac{\mu_0 nIR^2}{2r}\hat{\phi}$$

$$\mathbf{A}^M_{int}(r) = \mu_0 nI\frac{r}{2}\hat{\phi}$$

From the potential internal vector of Podolsky, taking the limit, we have:

$$\lim_{m\to\infty}\left(\frac{M}{m}I_1(mr) + \frac{w}{r}, \frac{FI_1(mr)}{m} + \frac{rB^p_z}{2} + \frac{j}{r}{,}\kappa\right) = \left(\frac{w}{r}, \frac{rB^p_z}{2} + \frac{j}{r}{,}\kappa\right) \tag{4.121}$$

Equating Maxwell's potential internal vector:

$$\left(0{,}\mu_0 nI\frac{r}{2}{,}0\right) = \left(\frac{w}{r}, \frac{rB^p_z}{2} + \frac{j}{r}{,}\kappa\right) \tag{4.122}$$

Therefore, we can conclude that:

$$w = 0 \tag{4.123}$$

$$\kappa = 0 \tag{4.124}$$

While the components in ϕ will only be equal if $j = 0$:

$$\frac{r\mu_0 nI}{2} = \frac{r\mu_0 nI}{2} + \frac{j}{r} \tag{4.125}$$

replacing the values of the constants, we have the potential internal vector for Podolsky:

$$\mathbf{A}^P_{int} = \left(\frac{M}{m}I_1(mr), \frac{FI_1(mr)}{m} + \frac{r\mu_0 nI}{2}, 0\right) \tag{4.126}$$

For the external case, in a similar way:

$$\lim_{m\to\infty}\left(-\frac{QK_1(mr)}{m} + \frac{s}{r}, -\frac{DK_1(mr)}{m} + \frac{f}{r}, \gamma\right) = \left(\frac{s}{r}, \frac{f}{r}, \gamma\right) \tag{4.127}$$

Equating the potential external vector for Maxwell:

$$\left(\frac{s}{r}, \frac{f}{r}, \gamma\right) = \left(0, \frac{I\mu_0 R^2}{2r}, 0\right) \tag{4.128}$$

$$s = 0 \tag{4.129}$$

$$\gamma = 0 \tag{4.130}$$

And then:

$$\frac{f}{r} = \frac{I\mu_0 R^2}{2r} \tag{4.131}$$

$$f = \frac{I\mu_0 R^2}{2} \tag{4.132}$$

Replacing the constants:

$$\mathbf{A}^P_{ext} = \left(-\frac{QK_1(mr)}{m}, -\frac{DK_1(mr)}{m} + \frac{I\mu_0 R^2}{2r}, 0\right) \tag{4.133}$$

Calculating the dimensional analysis for the magnetic potential vector in Maxwell, we found that $\mathbf{A}^M_{ext}(r) = \left[\frac{Tm^2}{m}\right] = [Tm]$, so, recalling:

$$\mu_0 = \left[\frac{Tm}{A}\right] \tag{4.134}$$

replacing Φ_P in $\mathbf{A}^P_{ext}(b)$:

$$\mathbf{A}^P_{ext}(b) = \frac{\mu_0 nIR^2}{2b}\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right) \tag{4.135}$$

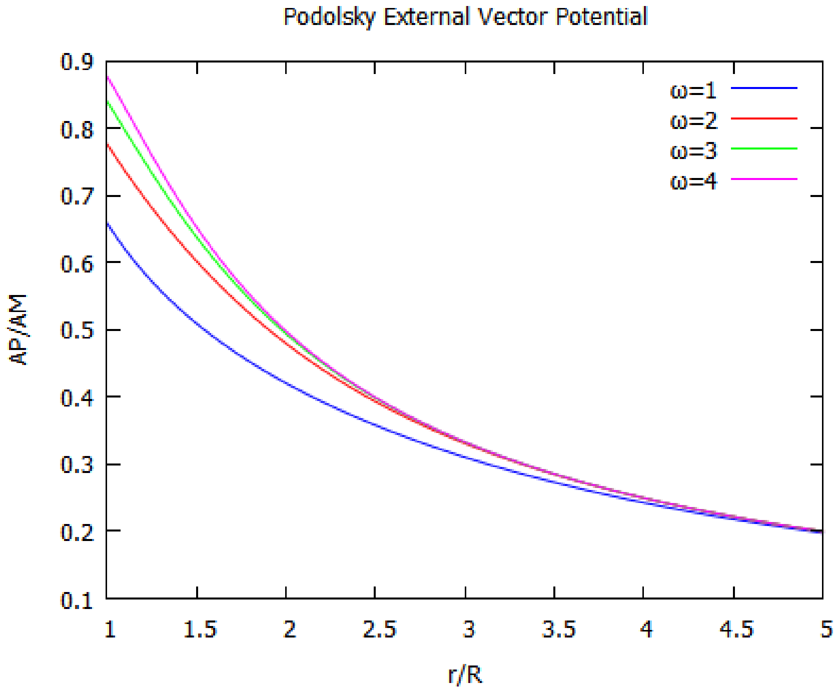
$$A^M \equiv \mu_0 nIR = [Tm] \tag{4.136}$$

$$\mathbf{A}^P_{ext}(b) = \frac{A^M R}{2b}\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right) \tag{4.137}$$

$$\mathbf{A}^P_{ext}(b) = \frac{A^M R}{2b}\left(1 - \frac{b}{mR}\frac{e^{-m(b-R)}}{\sqrt{Rb}}\right) \tag{4.138}$$

$$\frac{\mathbf{A}^P_{ext}}{A^M} = \frac{R}{2b} - \frac{e^{-m(b-R)}}{2m\sqrt{Rb}} \tag{4.139}$$

Figure 17 – Non dimensional external vector potential in Podolsky



Source: Author (2021).
Caption: External vector potential with ω values from 1 to 4.

Otherwise:

$$\mathbf{A}^P_{ext}(b) = \frac{\Phi^P_{ext}}{2\pi b} = \frac{\mu_0 n I R^2}{2b}\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right) \quad (4.140)$$

$$A^M \equiv \mu_0 n I R = [Tm] \quad (4.141)$$

$$\mathbf{A}^P_{ext}(b) = \frac{A^M R}{2b}\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right) \quad (4.142)$$

$$\chi \equiv \frac{b}{R} \quad (4.143)$$

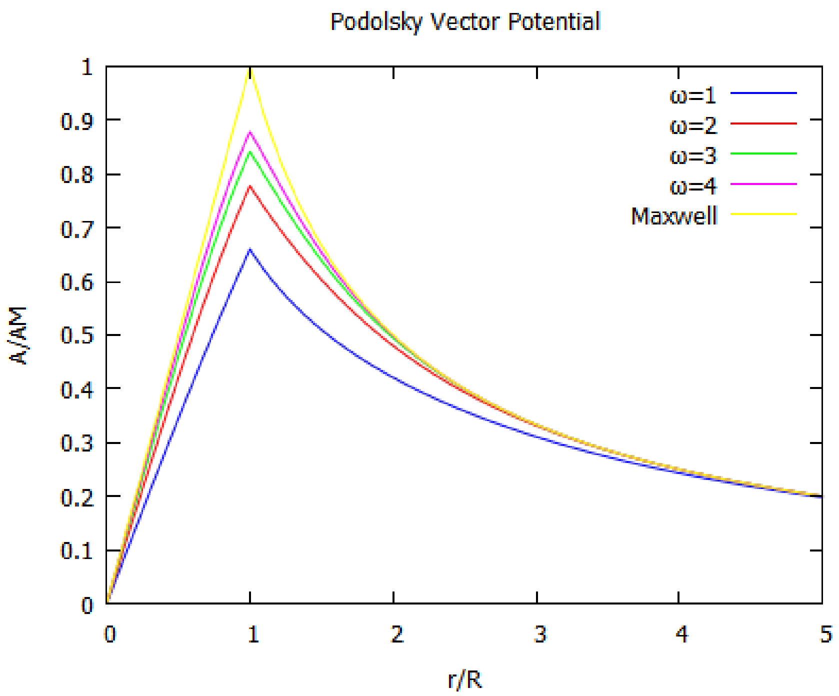
$$\omega \equiv mR \rightarrow m = \frac{\omega}{R} \quad (4.144)$$

$$\frac{\mathbf{A}^P_{ext}(\chi,\omega)}{A^M} = \frac{1}{2\chi}\left(1 - 2\chi I_1(\omega)K_1(\omega\chi)\right) \quad (4.145)$$

The graphical representation from the external vector potential in Podolsky, seen in Figure 17, show us that the higher value that ω assumes, the faster the vector potential will decay in the distance starting from the radius of the solenoid.

Combining both graphic results, the similarities with Maxwell reappears onde the ω values are greater, as shown in Figure 18.

Figure 18 – Non dimensional vector potential in Podolsky



Source: Author (2021).
Caption: Vector potential with ω values from 1 to 4.

4.4 BOUNDARY CONDITION

The boundary conditions of the problem are important to define the study region and how we will analyze the results originally obtained in the AB experiment and with our theoretical results. Fixing radius proportions and distance from the solenoid, we can observe that in the center or origin of the solenoid to the radius, where its turns separate the interior from the exterior we have the presence of the magnetic field and a magnetic potential that will vary according to the studied electrodynamics theory. The same occurs with the exterior and distances from the radius to infinity, but in these regions what is expected is that the field becomes null the greater the distance from the solenoid, and the magnetic potential presents a quantity that also varies according to the chosen theory. At the limit, when the radius is the exact border between the inner and outer regions, the field and the potential do not present continuity and their functions also show this discontinuity, corroborating the importance of fixing the boundary conditions. Considering r the radius and R the distance taken from the radius, when $r = R$, the internal and external fields must be equal, as well as the vector potential:

$$\mathbf{B}^P_{int} = \mathbf{B}^P_{ext} \tag{4.146}$$

$$\mathbf{A}^P_{int} = \mathbf{A}^P_{ext} \tag{4.147}$$

Equating the components, we have three equations:

$$FI_0(mR) + \mu_0 nI = DK_0(mR) \tag{4.148}$$

$$\frac{M}{m}I_1(mR) = -\frac{QK_1(mR)}{m} \tag{4.149}$$
$$\frac{FI_1(mR)}{m} + \frac{R\mu_0 nI}{2} = -\frac{DK_1(mR)}{m} + \frac{R\mu_0 nI}{2} \tag{4.150}$$

From equations 4.148 and 4.149:

$$FI_0(mR) + \mu_0 nI = DK_0(mR) \tag{4.151}$$
$$\frac{FI_1(mR)}{m} = -\frac{DK_1(mR)}{m} \tag{4.152}$$

Multiplying the second line by m:

$$FI_0(mR) + \mu_0 nI = DK_0(mR) \tag{4.153}$$
$$FI_1(mR) = -DK_1(mR) \tag{4.154}$$

Then, isolating F in the second equation:

$$FI_0(mR) + \mu_0 nI = DK_0(mR) \tag{4.155}$$
$$F = -\frac{DK_1(mR)}{I_1(mR)} \tag{4.156}$$

Replacing F in the first equation, we can find D:

$$\left(-\frac{DK_1(mR)}{I_1(mR)}\right) I_0(mR) + \mu_0 nI = DK_0(mR) \tag{4.157}$$

$$\mu_0 nI = \tfrac{I_1(mR)}{I_1(mR)} DK_0(mR) + I_0(mR)\left(\tfrac{DK_1(mR)}{I_1(mR)}\right) \tag{4.158}$$
$$\mu_0 nI = \tfrac{I_1(mR)DK_0(mR)+I_0(mR)DK_1(mR)}{I_1(mR)} \tag{4.159}$$
$$D = \tfrac{I_1(mR)\mu_0 nI}{[I_1(mR)K_0(mR)+I_0(mR)K_1(mR)]} \tag{4.160}$$

Using the proprieties of the functions:

$$I_\nu(x)K_{\nu+1}(x) + I_{\nu+1}(x)K_\nu(x) = \frac{1}{x} \tag{4.161}$$
$$I_\nu(mR)K_{\nu+1}(mR) + I_{\nu+1}(mR)K_\nu(mR) = \frac{1}{mR} \tag{4.162}$$

Then, D and F are:

$$D = mR\mu_0 nII_1(mR) \tag{4.163}$$

$$F = -mR\mu_0 nIK_1(mR) \tag{4.164}$$

We find two of the four constants in the problem, then, isolating M:

$$M = -\frac{QK_1(mR)}{I_1(mR)} \tag{4.165}$$

Replacing the constant M found in the magnetic field and the vector potential:

$$\mathbf{A}^P_{int} = \left(-\frac{QK_1(mR)}{mI_1(mR)} I_1(mr), \frac{FI_1(mr)}{m} + \frac{r\mu_0 nI}{2}, 0 \right) \tag{4.166}$$

$$\mathbf{A}^P_{ext} = \left(-\frac{QK_1(mr)}{m}, -\frac{DK_1(mr)}{m} + \frac{In\mu_0 R^2}{2r}, 0 \right) \tag{4.167}$$

$$\mathbf{B}^P_{int} = (0,0,FI_0(mr) + \mu_0 nI) \tag{4.168}$$

$$\mathbf{B}^P_{ext} = (0,0,DK_0(mr)) \tag{4.169}$$

being:

$$F = -mR\mu_0 nIK_1(mR) \tag{4.170}$$

$$D = mR\mu_0 nII_1(mR) \tag{4.171}$$

4.5 MAGNETIC FLUX

The magnetic flux is the measure of how much from the magnetic field is passing through some surface or object. Calculating in Podolsky, its results can corroborate the magnetic field and the vector potential, so we start from the inside of the solenoid and then to the outside.

4.5.1 Internal Flux

Calculation of the magnetic flux Φ_P of Podolsky, where we have the sum of the internal and external region for the whole field.

$$\Phi_P = \mathbf{B}_P \cdot d\mathbf{S} = \int_i \mathbf{B}^i_P \cdot d\mathbf{S} + \int_e \mathbf{B}^e_P \cdot d\mathbf{S} \tag{4.172}$$

Remembering that:

$$\mathbf{B}^i_P = (0,0,FI_0(mR) + \mu_0 nI) \tag{4.173}$$

$$\mathbf{B}^e_P = (0,0,DK_0(mR)) \tag{4.174}$$

Solving the first integral for the field :

$$\int_i \mathbf{B}^i_P \cdot d\mathbf{S} = \int_0^R \int_0^{2\pi} \int_0^{2\pi} (FI_0(mr) + \mu_0 nI)\, \hat{\mathbf{z}} \cdot rd\phi dr \hat{\mathbf{z}} \tag{4.175}$$

The recurrence relations of I_0 can be seen at the Appendix A.1. Then the internal flux is given by:

$$\Phi_P = 2\pi F \frac{1}{m} RI_1(mR) + \pi\mu_0 nIR^2 \tag{4.176}$$

4.5.2 External Flux

For the flux of the external field in Podolsky Φ_P^e, we have:

$$\int_e \mathbf{B}_P^e \cdot d\mathbf{S} = \int_R^b \int_0^{2\pi} DK_0(mR)\hat{\mathbf{z}} \cdot rd\phi dr\hat{\mathbf{z}} \tag{4.177}$$

$$= 2\pi D \int_R^b K_0(mR)rdr \tag{4.178}$$

From the recurrence relations and adding them, as seen in the Appendix, we have that the external flux is:

$$\Phi_{ext}^P = 2\pi D\frac{1}{m}RK_1(mR) - 2\pi D\frac{1}{m}bK_1(mb) \tag{4.179}$$

Therefore:

$$\Phi_P = \int_i \mathbf{B}_P^i \cdot d\mathbf{S} + \int_e \mathbf{B}_P^e \cdot d\mathbf{S} \tag{4.180}$$

$$= 2\pi F\frac{1}{m}RI_1(mR) + \pi\mu_0 nIR^2 + 2\pi D\frac{1}{m}RK_1(mR) - 2\pi D\frac{1}{m}bK_1(mb) \tag{4.181}$$

$$= 2\pi R\frac{1}{m}\left(FI_1(mR) + DK_1(mR)\right) + \pi\mu_0 nIR^2 - 2\pi D\frac{1}{m}bK_1(mb) \tag{4.182}$$

Remembering that $F = -mR\mu_0 nIK_1(mR)$ and $D = mR\mu_0 nII_1(mR)$ and replacing:

$$(FI_1(mR) + DK_1(mR)) = -mR\mu_0 nIK_1(mR)I_1(mR) + mR\mu_0 nII_1(mR)K_1(mR) \tag{4.183}$$

$$(FI_1(mR) + DK_1(mR)) = 0 \tag{4.184}$$

Replacing:

$$\Phi_{ext}^P = \pi\mu_0 nIR^2 - 2\pi D\frac{1}{m}bK_1(mb) \tag{4.185}$$

Replacing D:

$$\Phi_{ext}^P = \pi\mu_0 nIR^2 - 2\pi bR\mu_0 nII_1(mR)K_1(mb) \tag{4.186}$$

Remembering that the magnetic flux for Maxwell Φ_M:

$$\Phi_M = \pi\mu_0 nIR^2 \tag{4.187}$$

Therefore:

$$\Phi_{ext}^P = \Phi_M - 2b\frac{\Phi_M}{R}I_1(mR)K_1(mb) \tag{4.188}$$

$$= \Phi_M\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right) \tag{4.189}$$

Remembering that the external magnetic potential for Podolsky:

$$\mathbf{A}_{ext}^P = \left(0, - R\mu_0 nII_1(mR)K_1(mr) + \frac{In\mu_0 R^2}{2r}, 0\right) \tag{4.190}$$

The external vector potential for Podolsky for $r = b$:

$$\mathbf{A}^P_{ext}(b) = \left(0, -R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0 R^2}{2b}, 0\right) \tag{4.191}$$

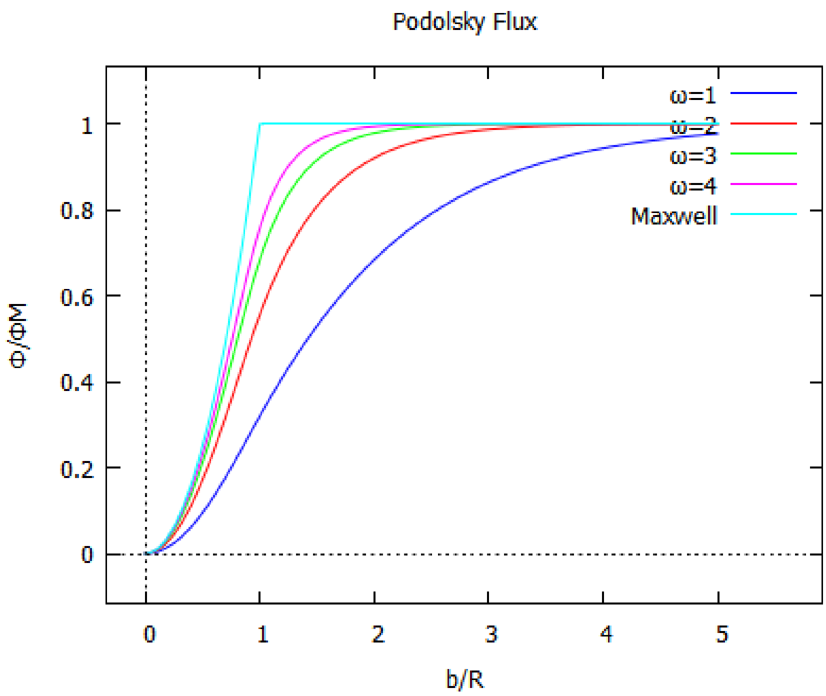
Isolating $\frac{1}{2\pi b}$:

$$\mathbf{A}^P_{ext}(b) = \left[0, \Phi^M\left(1 - 2\frac{b}{R}I_1(mR)K_1(mb)\right), 0\right] \tag{4.192}$$

$$\mathbf{A}^P_{ext}(b) = \left[0, \frac{\Phi^P_{ext}}{2\pi b}, 0\right] \tag{4.193}$$

Graphically, the Podolsky flux can be seen in Figure 19, where higher ω, the closest from Maxwell are the results. I grows from the center of the solenoid until the radius, then its continuous going further from the radius limit.

Figure 19 – Podolsky and Maxwell Flux



Source: Author (2021).

Caption: When the solenoid is charged, a magnetic field is generated in its radial direction.

4.6 SCHRÖDINGER'S EQUATION AND THE GENERALIZED GAUGE CONDITION

From the Schrödinger equation independent of time, without potential and considering symmetry in Laplacian:

$$\frac{1}{2m_e}\left[\frac{-\hbar^2}{b^2}\frac{d^2\psi}{d\phi^2} + q^2\mathbf{A}^2\psi + \frac{2i\hbar q}{b}\mathbf{A}\cdot\hat{\phi}\frac{d\psi}{d\phi} - \frac{\hbar q}{i}\psi\nabla\cdot\mathbf{A}\right] = E\psi \tag{4.194}$$

As from generalized Coulomb gauge[22], using $-2a = \frac{1}{m^2}$, we have that:

$$(1+\frac{1}{m^2}\nabla^2)\nabla\cdot\mathbf{A}=0 \tag{4.195}$$

$$\nabla\cdot\mathbf{A}+\frac{1}{m^2}\boldsymbol{\nabla^2}\left(\nabla\cdot\mathbf{A}\right)=\mathbf{0} \tag{4.196}$$

$$\nabla\cdot\mathbf{A}=-\frac{1}{m^2}\boldsymbol{\nabla^2}\left(\nabla\cdot\mathbf{A}\right) \tag{4.197}$$

Replacing the potential vector $\mathbf{A}^{P}_{ext}$:

$$\frac{1}{2m_e}\left\{\frac{-\hbar^2}{b^2}\frac{d^2\psi}{d\phi^2}+q^2\left[-\frac{QK_1(mb)}{m}\hat{r}-\left(R\mu_0 nII_1(mR)K_1(mb)+\frac{In\mu_0R^2}{2b}\right)\hat{\phi}+0\hat{z}\right]^2\psi+\right. \tag{4.198}$$

$$\frac{2i\hbar q}{b}\left[-\frac{QK_1(mb)}{m}\hat{r}-\left(R\mu_0 nII_1(mR)K_1(mb)+\frac{In\mu_0R^2}{2b}\right)\hat{\phi}+0\hat{z}\right]\cdot\hat{\phi}\frac{d\psi}{d\phi}-\frac{\hbar q}{i}\psi\left(\nabla\cdot\mathbf{A}\right)=E\psi$$

to

(4.200)

Solving $\nabla\cdot\mathbf{A}$:

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) \tag{4.201}$$

$$\nabla\cdot\mathbf{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{QK_1(mr)}{m}\right) \tag{4.202}$$

$$\nabla\cdot\mathbf{A}=-\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{QK_1(mr)}{m}\right)\right\} \tag{4.203}$$

$$=-\frac{QK_1'}{m}-\frac{QK_1}{rm} \tag{4.204}$$

From the derivative property we can find K_1'

$$K_{\nu-1}(x)+K_{\nu+1}(x)=-2K_\nu'(x) \tag{4.205}$$

$$\frac{K_0(mr)+K_2(mr)}{2}=\frac{-1}{m}K_1'(mr) \tag{4.206}$$

$$\nabla\cdot\mathbf{A}=\frac{Q\left(K_0(mr)+K_2(mr)\right)}{2}-\frac{QK_1(mr)}{rm}=0 \tag{4.207}$$

When using Coulomb gauge $\nabla\cdot\mathbf{A}=\mathbf{0}$, the term added can be removed, so we can deduce that $Q=0$. Solving $\mathbf{A}^2$:

$$\mathbf{A}^2=\left(R\mu_0 nII_1(mR)K_1(mb)+\frac{In\mu_0R^2}{2b}\right)^2 \tag{4.208}$$

Going back to the previous equation, we have that:

$$\frac{-\hbar^2}{b^2}\frac{d^2\psi}{d\phi^2} + q^2\left[\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right)^2\right]\psi + \frac{2i\hbar q}{b}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right)\frac{d\psi}{d\phi} = 2m_eE\psi \tag{4.209}$$

Multiplying the equation by $-\frac{b^2}{\hbar^2}$

$$\frac{d^2\psi}{d\phi^2} - \frac{2ibq}{\hbar}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right)\frac{d\psi}{d\phi} + \frac{b^2}{\hbar^2}q^2\left[-\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right)^2 + \frac{1}{q^2}2m_eE\right]\psi = 0 \tag{4.210}$$

Defining:

$$\alpha \equiv \frac{2ibq}{\hbar}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right) \tag{4.211}$$

$$\beta \equiv \frac{b^2}{\hbar^2}q^2\left[-\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right)^2 + \frac{1}{q^2}2m_eE\right] \tag{4.212}$$

So:

$$\frac{d^2\psi}{d\phi^2} - \alpha\frac{d\psi}{d\phi} + \beta\psi = 0 \tag{4.213}$$

$$y^2 - \alpha y + \beta = 0 \tag{4.214}$$

$$y = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \tag{4.215}$$

$$\psi = \psi_0 e^{y\phi} \tag{4.216}$$

Finding the roots:

$$y = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \tag{4.217}$$

$$\alpha^2 = \left(\frac{2ibq}{\hbar}R\mu_0 nII_1(mR)K_1(mb)\right)^2 + 2\left(\frac{2ibq}{\hbar}\mu_0 nI\right)^2 I_1(mR)K_1(mb)\frac{R^3}{2b} + \left(\frac{2iq}{\hbar}\frac{In\mu_0R^2}{2}\right)^2 \tag{4.218}$$

$$y = \frac{ibq}{\hbar}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0R^2}{2b}\right) \pm \frac{b}{\hbar}q\sqrt{\frac{Q^2K_1(mb)^2}{m^2} - \frac{2m_eE}{q^2}} \tag{4.219}$$

From the property bellow :

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x}K_\nu(x) \tag{4.220}$$

When $\nu = 1$ and $x = mb$:

$$\frac{K_0(mb) - K_2(mb)}{2} = -\frac{K_1(mb)}{mb} \tag{4.221}$$

We have:

$$y = \frac{ibq}{\hbar}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0 R^2}{2b}\right) \pm \frac{b}{\hbar}q\sqrt{\frac{Q^2K_1(mb)^2}{m^2} - \frac{2m_eE}{q^2}} \tag{4.222}$$

Then by the periodicity of the wave function, $\psi(\phi) = \psi(\phi + 2\pi)$:

$$\psi_0 e^{y\phi} = \psi_0 e^{y\phi + 2\pi y} \tag{4.223}$$

To satisfy the equation:

$$\frac{bq}{\hbar}\left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0 R^2}{2b}\right) \pm \frac{bq}{\hbar}\sqrt{\frac{2m_eE}{q^2} - \frac{Q^2K_1(mb)^2}{m^2}} = l \tag{4.224}$$

$$l = (0, \pm 1, \pm 2, ...)$$

$$E_l^P = \frac{q^2}{2m_e}\left[\frac{\hbar}{bq}n - \left(R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0 R^2}{2b}\right)\right]^2 + \frac{q^2}{2m_e}\frac{Q^2K_1(mb)^2}{m^2} \tag{4.225}$$

When $m \to \infty$, we have:

$$E_l^M = \frac{\hbar^2}{b^2}\frac{1}{2m_e}\left[l - \left(\frac{q}{\hbar}\frac{In\mu_0 R^2}{2}\right)\right]^2 \tag{4.226}$$

4.7 THE AHARONOV-BOHM EFFECT IN PODOLSKY ELECTRODYNAMICS

Since the magnetic field, flux and vector potential were calculated and defined, we can now apply the external potential vector to calculate the AB phase and get closer to an ending in this adventure in electrodynamic theories. Similarly to the calculations in Maxwell, we start from the potential external vector:

$$\mathbf{A}^P_{ext} = \left(-\frac{QK_1(mr)}{m}, -R\mu_0 nII_1(mR)K_1(mr) + \frac{In\mu_0 R^2}{2r}, 0\right) \tag{4.227}$$

and we can find the Aharonov-Bohm phase:

$$g(\mathbf{r}) \equiv \frac{q}{\hbar}\int_{\mathcal{O}}^{r} \mathbf{A}\cdot d\mathbf{r}' \tag{4.228}$$

Replacing, where $r = b$, where b is the distance to the wave function, therefore, we have that:

$$g(\mathbf{r}) = \frac{q}{\hbar}\int_{\mathcal{O}}^{r} \mathbf{A}^P_{ext}\cdot d\mathbf{r}' \tag{4.229}$$

$$g(\mathbf{r}) = \frac{q}{\hbar}\int_{\prime}^{2\pi}\left[\left(-\frac{QK_1(mb)}{m}\right)\hat{r} + \left(-R\mu_0 nII_1(mR)K_1(mb) + \frac{In\mu_0 R^2}{2b}\right)\hat{\phi} + 0\hat{z}\right]\cdot\hat{\phi}\,(\pm bd\phi) \tag{4.230}$$

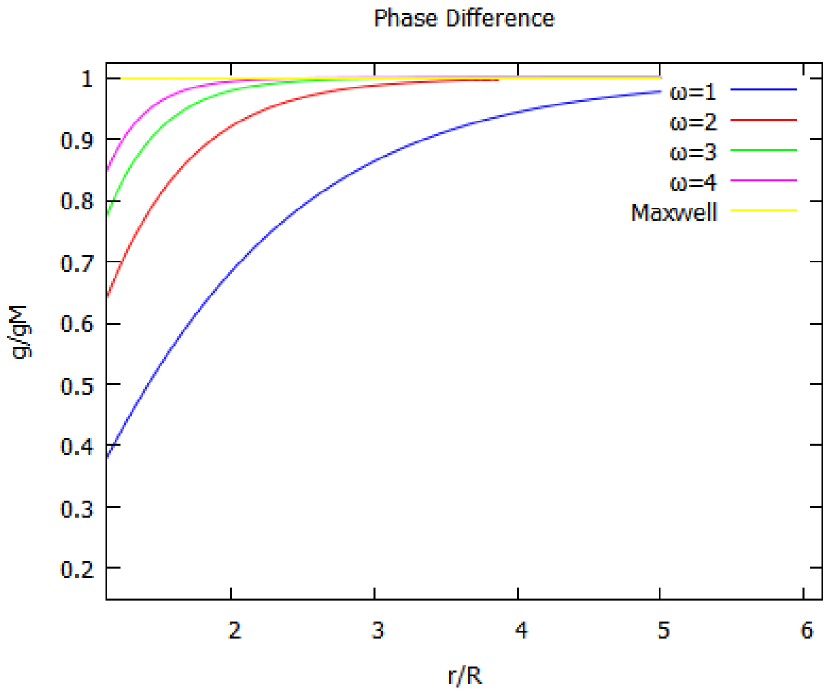
Solving the dot product:

$$\Delta g = \frac{q}{\hbar}\pi\left(-2Rb\mu_0 nII_1(mR)K_1(mb) + In\mu_0 R^2\right) \tag{4.231}$$

$$\Delta g = \frac{q}{\hbar}R^2\mu_0 nI\pi\left(-2\frac{b}{R}I_1(mR)K_1(mb) + 1\right) \tag{4.232}$$

Possessing now the Aharonov-Bohm phase we can write its behaviour graphically, seen in Figure 20, comparing with Maxwell results, which is a constant phase, while in Podolsky, it follows as previous results, higher the ω the closer from Maxwell.

Figure 20 – Podolsky and Maxwell Phase



Source: Author (2021).
Caption: Phase difference between Podolsky with ω values from 1 to 4 and Maxwell.

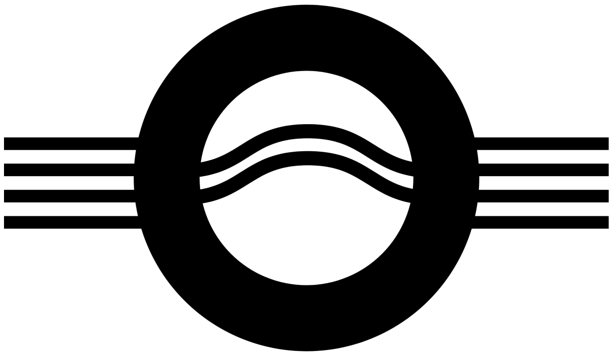
As seen in the graph, the greater the ratio of r to R, that is, the distance from the solenoid increases and the phase approaches the Maxwell result. Therefore, the AB phase differs from Maxwell only in the vicinity of the solenoid surface.

4.8 DIFFRACTION PATTERNS

If we could purpose to observe the Aharonov-Bohm effect in the Tonomura setup with Podolsky Electrodynamics, the image we would have seen in the patterns would be something similar to the image 21.

In the Fig. 21 we would expect to see the lines bending upwards or downwards in accordance with Eq. (4.232), but immediately close to the internal radius of the toroid,

Figure 21 – Exaggeration of the AB effect in Tonomura's experiment with Podolsky theory



Source: Author (2021).
Caption: The curve is an exaggeration of the effect and the lines were simplified for better comprehension.

with distances up to 10^{-15}m. As the toroid is coated with a superconductor, this effect tend to vanish by its interaction with the magnetic field. The rounded edges we see in the reffraction patterns in Tonomura's experiment hides this curve we would observe when studying the Aharanov-Bohm effect in Podolsky theory. Therefore, in order to apply the Tonomura experiment to test Podolsky electrodynamics, it would be necessary to redesign the experimental apparatus to probe distances very close to the solenoid.

5 FINAL CONSIDERATIONS

As known, Maxwell's equations are experimentally grounded and have long been consolidated, although it is not the most general theory possible for lack of fine-tuning. These adjustments can be made with theories of a higher order, which describe systems with greater degrees of complexity and include the classical laws of electromagnetism as a specific case of his theory, as in Podolsky's electrodynamics.

The Aharonov-Bohm experiment generated an important insight into how electromagnetism and quantum mechanics were related. When doing this study, the hypothesis elaborated by the researchers to answer because even without a field to cause alterations in the external region of study, in this case, the magnetically charged coil, the electron suffered alteration in its quantum phase and consequently in its energy. These results pointed to the magnetic potential generated together with the magnetic field that is formed in the solenoid when crossed by the current.

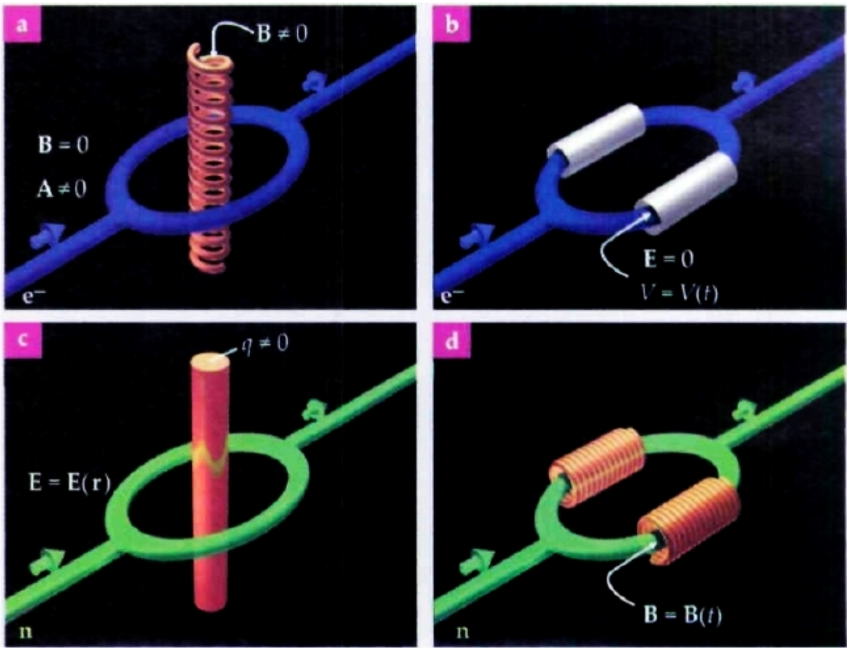
As demonstrated, Maxwell's results corroborate so that, even with a null external magnetic field, the magnetic vector potential exists only as a calculation tool without experimental proof of existence beyond the theoretical one; while in Podolsky we have more assertive theoretical results, even without experimental data to verify them, which seem to be correct due to the graphic behavior they present when compared with Maxwell's expected values, already validated.

In this thesis we show, for the first time, that the Aharonov-Bohm magnetic effect can be calculated by Podolsky's electrodynamics, confirming the invariance of the gauge of Maxwell's equations and his Lagrangian. The results show significant differences with respect to Maxwell, especially in the region close to the surface of the solenoid. In particular, our results show that the Aharonov-Bohm phase depends on the distance the beam passes from the surface of the solenoid, with the Maxwell result being recovered asymptotically over long distances. The same applies to the flux of the magnetic field.

Those results are obtained by the presence of higher-order derivatives in Podolsky, which softens the behavior of physical quantities. Not only is the magnetic field and the vector potential continuous at the interface between the inner and outer parts of the solenoid, but also their derivatives are continuous.

The next steps of this study would be to calculate the Aharonov-Casher electrical effect and a possible duality between these results, looking for a way to reach the magnetic results by the electrical effect and vice versa. In Figure 22 we can see in the first line the Aharonov-Bohm effect and in the second line the Aharonov-Casher effect.

Figure 22 – The experiments for Aharonov-Bohm and Aharonov-Casher



Source: [20].

Caption: In the Figure (a) is shown the Aharonov-Bohm magnetic effect, in Figure (b), the Aharonov-Bohm electric effect. Figure (c) shows the Aharonov-Casher electric effect and Figure (d) shows the magnetic Aharonov-Casher effect.

A RECURRENCE RELATIONS

The Helmholtz equation in cylindrical coordinates has the solution of the Bessel equation, wherein a differential form it has the solutions of Bessel and Neumann, $J_\nu(k\rho)$ and $N_\nu(k\rho)$, and their linear combinations; as well as the functions of Hankel $H^{(1)\nu}(k\rho)$ and $H^{(2)\nu}(k\rho)$ [21]. These Helmholtz equations describe the spatial part of the wave problem, but when we study the diffusion problem, the equation changes [21]. One of the solutions to this modified equation is the function $I_\nu(x)$, which can also be written in series, and has recurrence relations, used in this work.

$$I_\nu = e^{-\nu\pi\frac{i}{2}} J_\nu(xe^{\frac{i\pi}{2}}) \tag{a.1}$$

The other independent solution is given by the asymptotic behavior, $K_\nu(x)$, which can also be written in terms of the Henkel function $H^{(1)\nu}(k\rho)$ [21].

$$K_\nu \equiv \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) = \frac{\pi}{2} i^{\nu+1}[J_\nu(ix) + iN_\nu(ix)] \tag{a.2}$$

Both solutions have integral forms. These functions are often solutions to modified Bessel equations and are widely used in specific physical problems, such as diffusion problems [21].

The modified functions of Bessel $I_\nu(x)$ and $K_\nu(x)$ of order ν, which we used during this work, come from the solution of the modified Bessel equation, seen in equations **??** and a.4, the equation and then the solution [21].

$$x^2 y^" + xy' - (x^2 + \nu^2)y = 0 \tag{a.3}$$

$$y_n(x) = C_1 I_\nu(x) + C_2 K_\nu(x) \tag{a.4}$$

being I_ν and K_ν, where I_ν is a modified first type function and K_ν is a modified second type function, both being independent linear solutions. Thus, I_ν can be approximated as a growing exponential while K_ν, a decreasing exponential.

In the first section of Appendix A, we have the calculations that use the recurrence ratios of I_0 and then the recurrence ratios for K_0.

A.1 RECURRENCE RELATIONS OF I_0

From the recurrence relations of I_0, we have:

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I'_\nu(x) \tag{a.1}$$

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x) \tag{a.2}$$

Adding the equations, we have::

$$2I_{\nu-1}(x) = 2I'_\nu(x) + \frac{2\nu}{x} I_\nu(x) \tag{a.3}$$

$$xI_{\nu-1}(x) = x\frac{dI_\nu(x)}{dx} + \nu I_\nu(x) \tag{a.4}$$

When $\nu = 1$:

$$xI_0(x) = x\frac{dI_1(x)}{dx} + I_1(x) = \frac{d}{dx}(xI_1(x)) \tag{a.5}$$

Replacing $x = mr$:

$$mrI_0(mr) = \frac{1}{m}\frac{d}{dr}(mrI_1(mr)) \tag{a.6}$$

$$mrI_0(mr) = \frac{d}{dr}(rI_1(mr)) \tag{a.7}$$

$$rI_0(mr) = \frac{d}{mdr}(rI_1(mr)) \tag{a.8}$$

Substituting in the integral:

$$\int_i \mathbf{B}_P^i \cdot d\mathbf{S} = 2\pi F \int_0^R I_0(mr) r dr + \pi\mu_0 n I_{int} R^2 \tag{a.9}$$

$$= 2\pi F \int_0^R \frac{d}{mdr}(rI_1(mr))dr + \pi\mu_0 n I_{int} R^2 \tag{a.10}$$

$$= 2\pi F \frac{1}{m}\int_0^R d(rI_1(mr)) + \pi\mu_0 n I_{int} R^2 \tag{a.11}$$

$$\Phi_{int}^P = 2\pi F \frac{1}{m} R I_1(mR) + \pi\mu_0 n I_{int} R^2 \tag{a.12}$$

A.2 RECURRENCE RELATIONS OF K_0

From the recurrence relations of K_0, we have:

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x} K_\nu(x) \tag{a.1}$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2K'_\nu(x) \tag{a.2}$$

$$2K_{\nu-1}(x) = -\frac{2\nu}{x} K_\nu(x) - 2K'_\nu(x) \tag{a.3}$$

For $\nu = 1$ and $x = mr$:

$$K_0(mr) = -\frac{1}{mr} K_1(mr) - \frac{dK_1(mr)}{mdr} \tag{a.4}$$

Multiplying the equation by r:

$$rK_0(mr) = -\frac{1}{m} K_1(mr) - \frac{r}{m}\frac{dK_1(mr)}{dr} \tag{a.5}$$

$$rK_0(mr) = -\frac{1}{m}\left(K_1(mr) + r\frac{dK_1(mr)}{dr}\right) = -\frac{1}{m}\frac{d}{dr}(rK_1(mr)) \tag{a.6}$$

Replacing in integral, we have:

$$\int_e \mathbf{B}_P^e \cdot d\mathbf{S} = 2\pi D \int_R^b K_0(mR) r dr = -2\pi D \frac{1}{m} \int_R^b \frac{d}{dr}(rK_1(mr))dr \tag{a.7}$$

$$= -2\pi D \frac{1}{m} \int_R^b d(rK_1(mr)) \tag{a.8}$$

$$= 2\pi D \frac{1}{m} \left(RK_1(mR) - bK_1(mb)\right) \tag{a.9}$$

$$\Phi_{ext}^P = 2\pi D \frac{1}{m} RK_1(mR) - 2\pi D \frac{1}{m} bK_1(mb) \tag{a.10}$$

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